A Combo of Multivariable Third Degree Diophantine Equations

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Preface

A significant and important subject area of Theory of Numbers is the theory of Diophantine equations which concentrates on attempting to determine solutions in integers for higher degree and many parameters indeterminate equations. Obviously, polynomial Diophantine equations are many due to definition. Especially, the third degree Diophantine equation in two parameters falls into the theory of elliptic curves which is a developed theory. There are numerous motivating cubic equations with multiple variables which have kindled the interest among Mathematicians. For example, the representation of integers by binary cubic forms is known very little.

In this context, for simplicity and brevity, refer various forms of equations of degree three having many variables in [Carmichael.,1959, Dickson.,1952, Mordell.,1969, Gopalan et.al., 2015a, Gopalan et.al., 2015b, Premalatha, Gopalan et., 2020, Premalatha et.al., 2021, Shanthi, Gopalan.,2023, Thiruniraiselvi, Gopalan.,2021, Thiruniraiselvi, Gopalan., 2024a, Thiruniraiselvi, Gopalan., 2024b, Thiruniraiselvi et.al., 2024 Vidhyalakshmi, Gopalan., 2022a, Vidhyalakshmi, Gopalan., 2022b, Vidhyalakshmi, Gopalan., 2022c].

The focus in this book is on solving multivariable third degree Diophantine equations. These types of equations are significant since they concentrate on obtaining solutions in integers which satisfy the considered algebraic equations. These solutions play a vital role in different area of mathematics & science and help us in understanding the significance of number patterns.

This book contains a reasonable collection of cubic Diophantine equations with three, four, five and six unknowns. The procedure in obtaining varieties of solutions in integers for the polynomial Diophantine equations of degree three with three , four, five and six unknowns considered in this book are illustrated in an elegant manner.

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Chapter 1

NON - HOMOGENEOUS CUBIC DIOPHANTINE EQUATION WITH THREE PARAMETERS

1.1 Technical Procedure

The non-homogeneous ternary cubic equation under consideration is

$$x^2 + y^2 - xy = z^3 \tag{1.1}$$

Various choices of integer solutions to (1.1) are illustrated below:

Choice 1

The option

$$x = ky, k \geq 1 \tag{1.2}$$

in (1.1) gives

$$(k^2 - k + 1) y^2 = z^3$$

which is satisfied by

$$y = (k^2 - k + 1)\,\alpha^{3s}, z = (k^2 - k + 1)\,\alpha^{2s}, \alpha > 1, s \geq 0 \tag{1.3}$$

From (1.2) , we get

$$x = k(k^2 - k + 1)\,\alpha^{3s} \tag{1.4}$$

Thus , (1.3) & (1.4) satisfy (1.1).

Choice 2

The option

$$x = u + kz, y = u - kz, u \neq kz \quad (1.5)$$

in (1.1) gives

$$u^2 = z^2 (z - 3k^2) \quad (1.6)$$

The R.H.S. of (1.6) is a perfect square when

$$z = (s^2 + 3)\ k^2 \quad (1.7)$$

From (1.6) , we have

$$u = s(s^2 + 3) k^3 \quad (1.8)$$

Employing (1.7) & (1.8) in (1.5), it is seen that

$$\begin{aligned} x &= k^3 (s^2 + 3)(s + 1)\ , \\ y &= k^3 (s^2 + 3)(s - 1)\ , s > 1 \end{aligned} \quad (1.9)$$

Thus, (1.7) & (1.9) satisfy (1.1).

Note 1

The R.H.S. of (1.6) is also a perfect square for values of z given by

$$z_n = n^2 + 2kn + 4k^2 \quad (1.10)$$

From (1.6), we get

$$u_n = (n^2 + 2kn + 4k^2)\ (n + k) \quad (1.11)$$

In view of (1.5) , we have

$$\begin{aligned} x_n &= (n^2 + 2kn + 4k^2)\ (n + 2k)\ , \\ y_n &= (n^2 + 2kn + 4k^2)\ (n). \end{aligned} \quad (1.12)$$

Thus, (1.10) & (1.12) satisfy (1.1) .

Choice 3

The transformation

$$x = kz + v, y = kz - v, v \neq kz \qquad (1.13)$$

in (1.1) gives

$$v^2 = \frac{z^2 (z - k^2)}{3} \qquad (1.14)$$

The R.H.S. of (1.14) is a perfect square when

$$z = (3s^2 + 1) k^2 \qquad (1.15)$$

From (1.14), we get

$$v = s\ (3s^2 + 1)\ k^3 \qquad (1.16)$$

Using (1.15) & (1.16) in (1.13), we have

$$\begin{aligned} x &= k^3 (3s^2 + 1)\ (1 + s), \\ y &= k^3 (3s^2 + 1)\ (1 - s)\ , s \neq 1 \end{aligned} \qquad (1.17)$$

Thus, (1.15) & (1.17) satisfy (1.1).

Choice 4

Introduction of the transformations

$$x = u + v, y = u - v, u \neq v \qquad (1.18)$$

in (1.1) simplifies to

$$u^2 + 3v^2 = z^3 \qquad (1.19)$$

which is satisfied by

$$u = m(m^2 + 3n^2)\ , v = n(m^2 + 3n^2) \qquad (1.20)$$

and

$$z = m^2 + 3n^2 \qquad (1.21)$$

Substituting (1.20) in (1.18), we have

$$x = (m^2 + 3n^2)\,(m + n)\,,$$
$$y = (m^2 + 3n^2)\,(m - n)\,. \tag{1.22}$$

Thus, (1.21) & (1.22) satisfy (1.1).

Note 2

It is to be noted that (1.19) is also satisfied by

$$u = m^3 - 9mn^2\,, v = 3m^2 n - 3n^3\,, z = m^2 + 3n^2$$

For this choice, we have

$$x = m^3 - 9mn^2 + 3m^2 n - 3n^3\,,$$
$$y = m^3 - 9mn^2 - 3m^2 n + 3n^3\,,$$
$$z = m^2 + 3n^2\,.$$

Choice 5

The option

$$x = u + v, y = u - v, z = v, u \neq v \tag{1.23}$$

in (1.1) gives

$$u^2 = v^2\,(v - 3) \tag{1.24}$$

The R.H.S. of (1.24) is a perfect square when

$$v = s^2 + 3 \tag{1.25}$$

Using (1.25) in (1.24), we have

$$u = s\,(s^2 + 3)$$

From (1.23), the corresponding solutions to (1.1) are as below

$$x=(s^2+3)(s+1)\ ,$$
$$y=(s^2+3)(s-1)\ ,$$
$$z=(s^2+3)\ ,s\neq 1.$$

Note 3

The R.H.S. of (1.24) is a perfect square for values of v given by

$$v_n=n^2+2n+4$$

From (1.24) ,we get

$$u_n=(n^2+2n+4)(n+1)$$

From (1.23), the corresponding solutions to (1.1) are as below

$$x_n=(n^2+2n+4)(n+2),$$
$$y_n=(n^2+2n+4)(n),$$
$$z_n=(n^2+2n+4)\ ,n=1,2,3,...$$

Choice 6

The option

$$x=u+v, y=u-v, z=u, u\neq v \qquad (1.26)$$

in (1.1) gives

$$3\,v^2=u^2(u-1) \qquad (1.27)$$

The R.H.S. of (1.27) is a perfect square when

$$u=3s^2+1 \qquad (1.28)$$

Using (1.28) in 27, we have

$$v=s\ (3s^2+1)$$

From (1.26), the corresponding solutions to (1.1) are as below

$$x = (3s^2+1)(s+1) \text{ ,}$$
$$y = (3s^2+1)(1-s) \text{ ,}$$
$$z = (3s^2+1) \text{ ,} s \neq 1.$$

Choice 7

Treating (1.1) as a quadratic in x and solving for the same, we have

$$x = \frac{y \pm \sqrt{4z^3-3y^2}}{2} \qquad (1.29)$$

Let

$$\alpha^2 = 4z^3 - 3y^2 \qquad (1.30)$$

Assume

$$z = a^2 + 3b^2 \qquad (1.31)$$

Write the integer 4 in (1.30) as

$$4 = (1+i\sqrt{3})\,(1-i\sqrt{3}) \qquad (1.32)$$

Using (1.31) & (1.32) in (1.30) and employing factorization, consider

$$\alpha + i\sqrt{3}\,y = (1+i\sqrt{3})\,(a+i\sqrt{3}\,b)^3$$

On comparing, we get the values of α, y. From (1.29), the corresponding values to x are obtained.
For the benefit of readers, the two sets of integer solutions to (1.1) thus obtained are given below:

Set 1:
$$x = a^3 - 9ab^2 - 3a^2b + 3b^3 \text{ ,}$$
$$y = a^3 - 9ab^2 + 3a^2b - 3b^3 \text{ ,}$$
$$z = a^2 + 3b^2 \text{ .}$$

Set 2:

$$x = 6a^2 b - 6b^3,$$
$$y = a^3 - 9ab^2 + 3a^2 b - 3b^3,$$
$$z = a^2 + 3b^2.$$

Choice 8

Rewrite (1.30) as

$$\alpha^2 + 3y^2 = 4z^3 * 1 \tag{1.33}$$

The integer 1 in (1.33) is expressed as

$$1 = \frac{(3s^2 - 1 + i2s\sqrt{3})(3s^2 - 1 - i2s\sqrt{3})}{(3s^2+1)^2} \tag{1.34}$$

Substituting (1.31), (1.32) & (1.34) in (1.33) and employing factorization ,consider

$$\begin{aligned} \alpha + i\sqrt{3}y &= (1+i\sqrt{3})(a+i\sqrt{3}b)^3 \frac{(3s^2-1+i2s\sqrt{3})}{(3s^2+1)} \\ &= \frac{1}{(3s^2+1)}[F(s)+i\sqrt{3}G(s)][f(a,b)+i\sqrt{3}g(a,b)] \end{aligned} \tag{1.35}$$

where

$$\begin{aligned} F(s) &= (3s^2 - 1 - 6s), G(s) = (3s^2 - 1 + 2s) \\ f(a,b) &= (a^3 - 9ab^2), g(a,b) = (3a^2 b - 3b^3) \end{aligned} \tag{1.36}$$

From (1.35), we have

$$\begin{aligned} \alpha &= \frac{1}{(3s^2+1)}[F(s)\, f(a,b) - 3G(s)\, g(a,b)], \\ y &= \frac{1}{(3s^2+1)}[G(s)\, f(a,b) + F(s)\, g(a,b)]. \end{aligned} \tag{1.37}$$

As the thrust is on finding integer solutions, replacing a by $(3s^2+1)A$ and b by $(3s^2+1)B$ in (1.31) & (1.37) ,we have

$$z=(3s^2+1)^2(A^2+3B^2)\ ,$$
$$\alpha=(3s^2+1)^2[F(s)\ f(A,B)-3G(s)g(A,B)]\ , \qquad (1.38)$$
$$y=(3s^2+1)^2[G(s)\ f(A,B)+F(s)g(A,B)]\ ,$$

In view of (1.29), we have

$$x=\frac{(y\pm\alpha)}{2}$$
$$=\frac{(3s^2+1)^2}{2}\{f(A,B)[F(s)+G(s)]+g(A,B)[F(s)-3G(s)]\}\ ,$$
$$\frac{(3s^2+1)^2}{2}\{f(A,B)[G(s)-F(s)]+g(A,B)[F(s)+3G(s)]\}$$

After simplification using (1.36) ,we have

$$x=(3s^2+1)^2\{f(A,B)[3s^2-1-2s]+g(A,B)[-3s^2+1-6s]\}\ ,$$
$$(3s^2+1)^2\{f(A,B)[4s]+g(A,B)[6s^2-2]\} \qquad (1.39)$$

Thus, (1.1) is satisfied by (1.38) & (1.39).

Note 4

Apart from (1.34), one may have other representations to integer 1 which are exhibited below:

Representation 1:

$$1=\frac{(a(s)+i\ b(s)\sqrt{3})(a(s)-i\,b(s)\sqrt{3})}{(a(s)+1)^2}$$

where

$$a(s)=(6s^2-6s+1),b(s)=(2s-1)$$

Representation 2:

$$1=\frac{(a(s)+i\ b(s)\sqrt{3})(a(s)-i\,b(s)\sqrt{3})}{(a(s)+6s^2)^2}$$

where

$$a(s) = (r^2 - 3s^2), b(s) = 2rs$$

Representation 3:

$$1 = \frac{(a(s) + i\, b(s)\sqrt{3})(a(s) - i\, b(s)\sqrt{3})}{(a(s) + 2s^2)^2}$$

where

$$a(s) = (3r^2 - s^2), b(s) = 2rs$$

Representation 4:

$$1 = \frac{(1 + i\,\sqrt{3}\alpha_n)(1 - i\,\sqrt{3}\alpha_n)}{(\beta_n)^2} = \frac{(2 + i\, g_n)(2 - i\, g_n)}{(f_n)^2}$$

where

$$\alpha_n = \frac{1}{2\sqrt{3}} g_n = \frac{1}{2\sqrt{3}}[(2+\sqrt{3})^{n+1} - (2-\sqrt{3})^{n+1}],$$

$$\beta_n = \frac{1}{2} f_n = \frac{1}{2}[(2+\sqrt{3})^{n+1} + (2-\sqrt{3})^{n+1}], n = 0,1,2,...$$

A similar process, leads to four patterns of (1.1).

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Chapter 2

PEER SEARCH TO NON-UNIFORM THIRD DEGREE DIOPHANTINE EQUATION WITH THREE VARIABLES

2.1 Technical Procedure

The equation under consideration is

$$x^3+y^3+8k(x+y)=(2k+1)z^3, k\neq 0 \quad (2.1)$$

The option

$$x=u+v, y=u-v, z=2u, u\neq \pm v \quad (2.2)$$

in (2.1) gives

$$(8k+3)u^2-3v^2=8k \quad (2.3)$$

Again, taking

$$u=X+3T, v=X+(8k+3)T \quad (2.4)$$

in (2.3) ,it simplifies to the binary quadratic equation

$$X^2=(24k+9)T^2+1 \quad (2.5)$$

It is worth to mention that, if the value of k is taken as three times the Triangular number $\frac{n(n+1)}{2}$,then (2.5) reduces to

$$X^2=(6n+3)^2T^2+1$$

for which the solution is trivial. Therefore, choose k such that $24k+9$ is positive and square-free integer. Let (T_0, X_0) be the smallest positive integer solution to (5) ,a well-known pellian equation. After some algebra, the general solution (T_n, X_n) to (5) is given by

$$T_n = \frac{1}{2\sqrt{24k+9}} g_n , X_n = \frac{1}{2} f_n \tag{2.6}$$

where

$$f_n = (X_0 + \sqrt{24k+9}\, T_0)^{n+1} + (X_0 - \sqrt{24k+9}\, T_0)^{n+1},$$
$$g_n = (X_0 + \sqrt{24k+9}\, T_0)^{n+1} - (X_0 - \sqrt{24k+9}\, T_0)^{n+1}.$$

From (2.2) & (2.4), we have

$$\begin{aligned} x_n &= u_n + v_n = 2X_n + (8k+6)T_n \\ &= f_n + \frac{(4k+3)}{\sqrt{24k+9}} g_n , \\ y_n &= u_n - v_n = -8kT_n = -\frac{4k}{\sqrt{24k+9}} g_n , \\ z_n &= 2u_n = 2X_n + 6T_n = f_n + \frac{3}{\sqrt{24k+9}} g_n . \end{aligned} \tag{2.7}$$

Note 1

Apart from (2.4), take

$$u = X - 3T, v = X - (8k+3)T$$

In this case, (2.1) is satisfied by

$$\begin{aligned} x_n &= u_n + v_n = 2X_n - (8k+6)T_n \\ &= f_n - \frac{(4k+3)}{\sqrt{24k+9}} g_n , \\ y_n &= u_n - v_n = 8kT_n = \frac{4k}{\sqrt{24k+9}} g_n , \\ z_n &= 2u_n = 2X_n - 6T_n = f_n - \frac{3}{\sqrt{24k+9}} g_n . \end{aligned}$$

To analyze the nature of solutions, one has to take special value to k in (2.5).

Illustration 1

Considering k=2 in (2.5) , it is written as

$$X^2 = 57\ T^2 + 1 \tag{2.8}$$

whose smallest positive integer solution is

$$T_0 = 20, X_0 = 151$$

From (2.6) ,the general solution to (2.8) are given by

$$T_n = \frac{1}{2\sqrt{57}} g_n, X_n = \frac{1}{2} f_n$$

where

$$f_n = (151 + 20\sqrt{57})^{n+1} + (151 + 20\sqrt{57})^{n+1},$$
$$g_n = (151 + 20\sqrt{57})^{n+1} - (151 + 20\sqrt{57})^{n+1},$$

From (2.7), the corresponding integer solutions to the equation

$$x^3 + y^3 + 16(x + y) = 5z^3 \tag{2.9}$$

are given by

$$\begin{aligned} x_n &= f_n + \frac{11}{\sqrt{57}} g_n, \\ y_n &= -\frac{8}{\sqrt{57}} g_n, \\ z_n &= f_n + \frac{3}{\sqrt{57}} g_n, n = 0,1,2,... \end{aligned} \tag{2.10}$$

The recurrence relations satisfied by the solutions to (2.9) given by (2.10) are presented below:

$$x_{n+2} - 302x_{n+1} + x_n = 0,$$
$$y_{n+2} - 302y_{n+1} + y_n = 0,$$
$$z_{n+2} - 302z_{n+1} + z_n = 0, \; n = 0,1,2,...$$

A few numerical solutions to (2.9) are shown below:

$$x_0 = 742, y_0 = -320, z_0 = 422$$
$$x_1 = 224082, y_1 = -96640, z_1 = 127442$$
$$x_2 = 67672022, y_2 = -29184960, z_2 = 38487062$$

Remarkable observations

1. $8x_{2n+1} + 11y_{2n+1} + 16$ is two times a square integer
2. $3(8z_{2n+1} + 3y_{2n+1} + 16)$ is a square multiple of 6
3. Each of the following expressions is a cubical integer
   $8x_{3n+2} + 11y_{3n+2} + 3(8x_n + 11y_n)$,
   $8z_{3n+2} + 3y_{3n+2} + 3(8z_n + 3y_n)$
4. Each of the following equations represents hyperbola

$$(8x_n + 11y_n)^2 - 57(x_n - z_n)^2 = 256,$$
$$(8z_n + 3y_n)^2 - 57(x_n - z_n)^2 = 256,$$
$$(8x_n + 11y_n)^2 - 57\,y_n^{\,2} = 256,$$
$$(8z_n + 3y_n)^2 - 57y_n^{\,2} = 256.$$

5. Each of the following equations represents parabola
   $8(8x_{2n+1} + 11y_{2n+1} + 16) - 57y_n^{\,2} = 256,$
   $8(8z_{2n+1} + 3y_{2n+1} + 16) - 57(x_n - z_n)^2 = 256.$
6. $16x_{4n+3} + 22y_{4n+3} + 32(8x_n + 11y_n)^2 - 32$ is a quartic integer
7. Formulation of Second order Ramanujan numbers :

   From each of the solutions of (2.9), one can find Second order Ramanujan numbers with base numbers as real integers.

   Illustration

$$y_0 = -320$$

$$= 1*(-320) = 2*(-160) = (-4)*80 = (-5)*64 = 8*(-40) = (-10)*32 = 20*(-16)$$

$$= \quad F_1 \quad F_2 \quad F_3 \quad F_4 \quad F_5 \quad F_6 \quad F_7$$

$$F_1 = F_2 \Rightarrow (1-320)^2 + (2+160)^2 = (1+320)^2 + (2-160)^2$$

$$= 319^2 + 162^2 = 321^2 + 158^2 = 128005$$

$$F_1 = F_3 \Rightarrow (1-320)^2 + (-4-80)^2 = (1+320)^2 + (-4+80)^2$$

$$= 319^2 + 84^2 = 321^2 + 76^2 = 108817$$

$$F_1 = F_4 \Rightarrow (1-320)^2 + (-5-64)^2 = (1+320)^2 + (-5+64)^2$$

$$= 319^2 + 69^2 = 321^2 + 59^2 = 106522$$

$$F_1 = F_5 \Rightarrow (1-320)^2 + (8+40)^2 = (1+320)^2 + (8-40)^2$$

$$= 319^2 + 48^2 = 321^2 + 32^2 = 104065$$

$$F_1 = F_6 \Rightarrow (1-320)^2 + (-10-32)^2 = (1+320)^2 + (-10+32)^2$$

$$= 319^2 + 42^2 = 321^2 + 22^2 = 103525$$

$$F_1 = F_7 \Rightarrow (1-320)^2 + (20+16)^2 = (1+320)^2 + (20-16)^2$$

$$= 319^2 + 36^2 = 321^2 + 4^2 = 103057$$

$$F_2 = F_3 \Rightarrow (2-160)^2 + (-4-80)^2 = (2+160)^2 + (-4+80)^2$$

$$= 158^2 + 84^2 = 162^2 + 76^2 = 32020$$

$$F_2 = F_4 \Rightarrow (2-160)^2 + (-5-64)^2 = (2+160)^2 + (-5+64)^2$$
$$= 158^2 + 69^2 = 162^2 + 59^2 = 29725$$
$$F_2 = F_5 \Rightarrow (2-160)^2 + (8+40)^2 = (2+160)^2 + (8-40)^2$$
$$= 158^2 + 48^2 = 162^2 + 32^2 = 27268$$
$$F_2 = F_6 \Rightarrow (2-160)^2 + (-10-32)^2 = (2+160)^2 + (-10+32)^2$$
$$= 158^2 + 42^2 = 162^2 + 22^2 = 26728$$
$$F_2 = F_7 \Rightarrow (2-160)^2 + (20+16)^2 = (2+160)^2 + (20-16)^2$$
$$= 158^2 + 36^2 = 162^2 + 4^2 = 26260$$
$$F_3 = F_4 \Rightarrow (-4+80)^2 + (-5-64)^2 = (-4-80)^2 + (-5+64)^2$$
$$= 76^2 + 69^2 = 84^2 + 59^2 = 10537$$
$$F_3 = F_5 \Rightarrow (-4+80)^2 + (8+40)^2 = (-4-80)^2 + (8-40)^2$$
$$= 76^2 + 48^2 = 84^2 + 32^2 = 8080$$
$$F_3 = F_6 \Rightarrow (-4+80)^2 + (-10-32)^2 = (-4-80)^2 + (-10+32)^2$$
$$= 76^2 + 42^2 = 84^2 + 22^2 = 7540$$
$$F_3 = F_7 \Rightarrow (-4+80)^2 + (20+16)^2 = (-4-80)^2 + (20-16)^2$$
$$= 76^2 + 36^2 = 84^2 + 4^2 = 7072$$
$$F_4 = F_5 \Rightarrow (-5+64)^2 + (8+40)^2 = (-5-64)^2 + (8-40)^2$$
$$= 59^2 + 48^2 = 69^2 + 32^2 = 5785$$
$$F_4 = F_6 \Rightarrow (-5+64)^2 + (-10-32)^2 = (-5-64)^2 + (-10+32)^2$$
$$= 59^2 + 42^2 = 69^2 + 22^2 = 5245$$
$$F_4 = F_7 \Rightarrow (-5+64)^2 + (20+16)^2 = (-5-64)^2 + (20-16)^2$$
$$= 59^2 + 36^2 = 69^2 + 4^2 = 4777$$
$$F_4 = F_5 \Rightarrow (-5+64)^2 + (8+40)^2 = (-5-64)^2 + (8-40)^2$$
$$= 59^2 + 48^2 = 69^2 + 32^2 =$$

$$F_4 = F_5 \Rightarrow (-5+64)^2 + (8+40)^2 = (-5-64)^2 + (8-40)^2$$
$$= 59^2 + 48^2 = 69^2 + 32^2 = 5785$$
$$F_4 = F_6 \Rightarrow (-5+64)^2 + (-10-32)^2 = (-5-64)^2 + (-10+32)^2$$
$$= 59^2 + 42^2 = 69^2 + 22^2 = 5245$$
$$F_4 = F_7 \Rightarrow (-5+64)^2 + (20+16)^2 = (-5-64)^2 + (20-16)^2$$
$$= 59^2 + 36^2 = 69^2 + 4^2 = 4777$$
$$F_5 = F_6 \Rightarrow (8-40)^2 + (-10-32)^2 = (8+40)^2 + (-10+32)^2$$
$$= 32^2 + 42^2 = 48^2 + 22^2 = 2788$$
$$F_5 = F_7 \Rightarrow (8-40)^2 + (20+16)^2 = (8+40)^2 + (20-16)^2$$
$$= 32^2 + 36^2 = 48^2 + 4^2 = 2320$$
$$F_6 = F_7 \Rightarrow (-10+32)^2 + (20+16)^2 = (-10-32)^2 + (20-16)^2$$
$$= 22^2 + 36^2 = 42^2 + 4^2 = 1780$$

Thus, 128005, 108817, 106522, 104065, 103525, 103057, 32020, 29725, 27268, 26728, 26260, 10537, 8080, 7540, 7072, 5785, 5245, 4777, 2788, 2320, 1780 represent second order Ramanujan numbers with base numbers as real integers.

Illustration 2

Considering k=4 in (2.5), it is written as

$$X^2 = 105\, T^2 + 1 \qquad (2.11)$$

whose smallest positive integer solution is

$$T_0 = 4, X_0 = 41$$

From (2.6) the general solution to (2.11) are given by

$$T_n = \frac{1}{2\sqrt{105}} g_n \,, X_n = \frac{1}{2} f_n$$

where

$$f_n = (41+4\sqrt{105})^{n+1} + (41+4\sqrt{105})^{n+1},$$
$$g_n = (41+4\sqrt{105})^{n+1} - (41+4\sqrt{105})^{n+1},$$

From (2.7), the corresponding integer solutions to the equation

$$x^3 + y^3 + 32(x+y) = 9z^3 \qquad (2.12)$$

are given by

$$\begin{aligned} x_n &= f_n + \frac{19}{\sqrt{105}} g_n, \\ y_n &= -\frac{16}{\sqrt{105}} g_n, \\ z_n &= f_n + \frac{3}{\sqrt{105}} g_n, \; n = 0,1,2,... \end{aligned} \qquad (2.13)$$

The recurrence relations satisfied by the solutions to (2.12) given by (2.13) are presented below:

$$\begin{aligned} x_{n+2} - 82x_{n+1} + x_n &= 0, \\ y_{n+2} - 82y_{n+1} + y_n &= 0, \\ z_{n+2} - 82z_{n+1} + z_n &= 0, \; n = 0,1,2,... \end{aligned}$$

A few numerical solutions to (2.12) are shown below:

$$\begin{aligned} & x_0 = 234, y_0 = -128, z_0 = 106 \\ & x_1 = 19186, y_1 = -10496, z_1 = 8690 \\ & x_2 = 1573018, y_2 = -860544, z_2 = 712474 \end{aligned}$$

Remarkable observations

1. The expressions below represent Nasty Numbers.

$$\mathbf{96x_{2n+1} + 114y_{2n+1} + 192}$$
$$\mathbf{96z_{2n+1} + 18y_{2n+1} + 192}$$
$$\mathbf{114z_{2n+1} - 18x_{2n+1} + 192}$$

2. The expressions below represent twice a cubical integer

$$16x_{3n+2} + 19y_{3n+2} + 3(16x_n + 19y_n)$$

$$16x_{3n+2} + 19y_{3n+2} + 3(16z_n + 3y_n)$$

$$16x_{3n+2} + 19y_{3n+2} + 3(19z_n - 3x_n)$$

3. Each of the following expressions is a quintic integer

$$(16x_n + 19y_n)^2 - 105\, y_n{}^2$$

$$(16z_n + 3y_n)^2 - 105y_n{}^2$$

$$(19z_n - 3x_n)^2 - 105y_n{}^2$$

$$16(16x_{2n+1} + 19y_{2n+1} + 32) - 105y_n{}^2$$

$$16(16z_{2n+1} + 3y_{2n+1} + 32) - 105y_n{}^2$$

$$16(19z_{2n+1} - 3x_{2n+1} + 32) - 105y_n{}^2$$

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Chapter 3

TECHNIQUE TO SOLVE NON-HOMOGENEOUS TERNARY CUBIC EQUATION

3.1 Technical Procedure

The non-homogeneous ternary cubic equation for obtaining integer solutions under consideration is

$$(x+y)(x^2-4xy+y^2)+4z^2=0 \tag{3.1}$$

The substitution of the linear transformations

$$x=u+v,\ y=u-v,\ z=2^n k u,\ u\neq v\neq 0,\ n\geq 1,\ k>0 \tag{3.2}$$

in (3.1) leads to the pellian equation

$$Y^2=3v^2+(2^{2n-1}k^2)^2 \tag{3.3}$$

where

$$Y=u-2^{2n-1}k^2 \tag{3.4}$$

The smallest positive integer solutions to (3.3) are given by

$$v_0=2^{2n-1}k^2,\ Y_0=2^{2n}k^2 \tag{3.5}$$

To obtain the other solutions to (3.3),consider the pellian equation

$$Y^2=3v^2+1$$

whose general solution is given by

$$\tilde{Y}_s=\frac{1}{2}f_s,\ \tilde{v}_s=\frac{1}{2\sqrt{3}}g_s \tag{3.6}$$

where

$$f_s=(2+\sqrt{3})^{s+1}+(2-\sqrt{3})^{s+1},$$
$$g_s=(2+\sqrt{3})^{s+1}-(2-\sqrt{3})^{s+1}.$$

Employing the lemma of Brahmagupta between the solutions (3.5) & (3.6) ,we have

$$v_{s+1} = v_0 \tilde{Y}_s + Y_0 \tilde{v}_s = 2^{2n-2} k^2 f_s + \frac{2^{2n-1}}{\sqrt{3}} k^2 g_s ,$$

$$Y_{s+1} = Y_0 \tilde{Y}_s + 3 v_0 \tilde{v}_s = 2^{2n-1} k^2 f_s + 2^{2n-2} k^2 g_s \sqrt{3} , s = -1,0,1,...$$

In view of (3.4) ,we get

$$u_{s+1} = 2^{2n-1} k^2 + 2^{2n-1} k^2 f_s + 2^{2n-2} k^2 g_s \sqrt{3}$$

From (3.2), we have

$$x_{s+1}(k,s,n) = 2^{2n-1} k^2 + 3*2^{2n-2} k^2 f_s + \frac{5*2^{2n-2}}{\sqrt{3}} k^2 g_s ,$$

$$y_{s+1}(k,s,n) = 2^{2n-1} k^2 + 2^{2n-2} k^2 f_s + \frac{2^{2n-2}}{\sqrt{3}} k^2 g_s ,$$

$$z_{s+1}(k,s,n) = 2^n k [2^{2n-1} k^2 + 2^{2n-1} k^2 f_s + \sqrt{3}\, 2^{2n-2} k^2 g_s], s = -1,0,1,...$$

A few numerical solutions to (3.1) are given in Table 3.1 below:

Table 3.1-Numerical solutions

s	$x_{s+1}(k,s,n)$	$y_{s+1}(k,s,n)$	$z_{s+1}(k,s,n)$
-1	$2^{2n+1} k^2$	$2^{2n} k^2$	$3*2^{3n-1} k^3$
0	$3*2^{2n+1} k^2$	$2^{2n+1} k^2$	$2^{3n+2} k^3$
1	$21*2^{2n} k^2$	$3*2^{2n+1} k^2$	$54*2^{3n-2} k^3$

The recurrence relations satisfied by the solutions of (3.1) are given below:

$$x_{s+3}(k,s,n) - 4 x_{s+2}(k,s,n) + x_{s+1}(k,s,n) = -2^{2n} k^2$$

$$y_{s+3}(k,s,n) - 4 y_{s+2}(k,s,n) + y_{s+1}(k,s,n) = -2^{2n} k^2$$

$$z_{s+3}(k,s,n) - 4 z_{s+2}(k,s,n) + z_{s+1}(k,s,n) = -2^{3n} k^3$$

Interesting relations among the integer solutions to (3.1)

(i) $x_{s+1}(k,s,n) = y_{s+2}(k,s,n)$

(ii) $[x_{s+1}(k,s,n) + y_{s+1}(k,s,n) - 2^{2n} k^2]^2 - 3[x_{s+1}(k,s,n) - y_{s+1}(k,s,n)]^2 = (2^n k)^4$, a quartic integer

(iii) $2[3*2^{n+1} k\, y_{2s+2}(k,s,n) - 2 z_{2s+2}(k,s,n) - (2^n k)^3]$ is a square multiple of $(2^n k)^3$

(iv) $2[10 z_{2s+2} - 3*2^{n+1} k\, x_{2s+2}(k,s,n) - (2^n k)^3]$ is a square multiple of $(2^n k)^3$

(v) $2[5y_{2s+2}(k,s,n)-x_{2s+2}(k,s,n)-(2^n k)^2]$ is a square multiple of $(2^n k)^2$

Each of the following expressions in (vi), (vii) & (viii) is a cubical integer:

(vi) $2^{n+1}k[5y_{3s+3}(k,s,n)-x_{3s+3}(k,s,n)+15y_{s+1}(k,s,n)-3x_{s+1}(k,s,n)-2^{2n+3}k^2]$

(vii)

$$2^2[5z_{3s+3}(k,s,n)-3*2^n k x_{3s+3}(k,s,n)+15z_{s+1}(k,s,n)-9*2^n k x_{s+1}(k,s,n)-2^{3n+2}k^3]$$

(viii)

$$2^2[3*2^n k y_{3s+3}(k,s,n)-z_{3s+3}(k,s,n)-3z_{s+1}(k,s,n)+9*2^n k y_{s+1}(k,s,n)-2^{3n+2}k^3]$$

(ix)

$$(5y_{s+1}(k,s,n)-x_{s+1}(k,s,n)-2^{2n+1}k^2)^2-3(x_{s+1}(k,s,n)-3y_{s+1}(k,s,n)+2^{2n}k^2)^2=(2^n k)^4$$

(x)

$$2^2[5z_{s+1}(k,s,n)-3*2^n k x_{s+1}(k,s,n)-2^{3n}k^3]^2-12[z_{s+1}(k,s,n)-2^{n+1}k y_{s+1}(k,s,n)-2^{3n-1}k^3+2^{3n}k^3]^2$$
$$=(2^n k)^6$$

(xi)

$$2^2[-z_{s+1}(k,s,n)+3*2^n k y_{s+1}(k,s,n)-2^{3n}k^3]^2-12[z_{s+1}(k,s,n)-2^{n+1}k y_{s+1}(k,s,n)-2^{3n-1}k^3+2^{3n}k^3]^2$$
$$=(2^n k)^6$$

Formulation of Second order Ramanujan numbers :

From each of the solutions of (3.1), one can find Second order Ramanujan numbers with base numbers as real integers.

Illustration 3.1

Consider from Table 3.1

$$z_0(2,-1,1) = 96 = 1*96 = 2*48 = 3*32 = 4*24 = 6*16 = 8*12$$

$$= \quad F_1 \qquad F_2 \qquad F_3 \qquad F_4 \qquad F_5 \quad = \quad F_6$$

$$F_1 = F_2 \Rightarrow (96+1)^2 + (48-2)^2 = (96-1)^2 + (48+2)^2$$

$$= 97^2 + 46^2 = 95^2 + 50^2 = 11525$$

$$F_1 = F_3 \Rightarrow (96+1)^2 + (32-3)^2 = (96-1)^2 + (32+3)^2$$

$$= 97^2 + 29^2 = 95^2 + 35^2 = 10250$$

$$F_1 = F_4 \Rightarrow (96+1)^2 + (24-4)^2 = (96-1)^2 + (24+4)^2$$

$$= 97^2 + 20^2 = 95^2 + 28^2 = 9809$$

$$F_1 = F_5 \Rightarrow (96+1)^2 + (16-6)^2 = (96-1)^2 + (16+6)^2$$

$$= 97^2 + 10^2 = 95^2 + 22^2 = 9509$$

$$F_1 = F_6 \Rightarrow (96+1)^2 + (12-8)^2 = (96-1)^2 + (12+8)^2$$

$$= 97^2 + 4^2 = 95^2 + 20^2 = 9425$$

$$F_2 = F_3 \Rightarrow (48+2)^2 + (32-3)^2 = (48-2)^2 + (32+3)^2$$

$$= 50^2 + 29^2 = 46^2 + 35^2 = 3341$$

$$F_2 = F_4 \Rightarrow (48+2)^2 + (24-4)^2 = (48-2)^2 + (24+4)^2$$

$$= 50^2 + 20^2 = 46^2 + 28^2 = 2900$$

$$F_2 = F_5 \Rightarrow (48+2)^2 + (16-6)^2 = (48-2)^2 + (16+6)^2$$
$$= 50^2 + 10^2 = 46^2 + 22^2 = 2600$$
$$F_2 = F_6 \Rightarrow (48+2)^2 + (12-8)^2 = (48-2)^2 + (12+8)^2$$
$$= 50^2 + 4^2 = 46^2 + 20^2 = 2516$$
$$F_3 = F_4 \Rightarrow (32+3)^2 + (24-4)^2 = (32-3)^2 + (24+4)^2$$
$$= 35^2 + 20^2 = 29^2 + 28^2 = 1625$$
$$F_3 = F_5 \Rightarrow (32+3)^2 + (16-6)^2 = (32-3)^2 + (16+6)^2$$
$$= 35^2 + 10^2 = 29^2 + 22^2 = 1325$$
$$F_3 = F_6 \Rightarrow (32+3)^2 + (12-8)^2 = (32-3)^2 + (12+8)^2$$
$$= 35^2 + 4^2 = 29^2 + 20^2 = 1241$$
$$F_4 = F_5 \Rightarrow (24+4)^2 + (16-6)^2 = (24-4)^2 + (16+6)^2$$
$$= 28^2 + 10^2 = 20^2 + 22^2 = 884$$
$$F_5 = F_6 \Rightarrow (16+6)^2 + (12-8)^2 = (16-6)^2 + (12+8)^2$$
$$= 22^2 + 4^2 = 10^2 + 20^2 = 500$$

Thus , 11525, 10250, 9809, 9509, 9425, 3341, 2900, 2600, 2516, 1625, 1325, 1241, 884, 500 represent second order Ramanujan numbers with base numbers as real integers.

Note 3.1

In illustration 3.1, the factors in each of F_2, F_4, F_5, F_6 belong to the same parity. In this case, there is an another way of obtaining second order Ramanujan numbers. The process of getting the same is illustrated below:

$$F_2 = F_4 \Rightarrow 25^2 - 23^2 = 14^2 - 10^2$$
$$\Rightarrow 25^2 + 10^2 = 14^2 + 23^2 = 725$$
$$F_2 = F_5 \Rightarrow 25^2 - 23^2 = 11^2 - 5^2$$
$$\Rightarrow 25^2 + 5^2 = 11^2 + 23^2 = 650$$
$$F_2 = F_6 \Rightarrow 25^2 - 23^2 = 10^2 - 2^2$$
$$\Rightarrow 25^2 + 2^2 = 10^2 + 23^2 = 629$$
$$F_4 = F_5 \Rightarrow 14^2 - 10^2 = 11^2 - 5^2$$
$$\Rightarrow 14^2 + 5^2 = 11^2 + 10^2 = 221$$
$$F_5 = F_6 \Rightarrow 11^2 - 5^2 = 10^2 - 2^2$$
$$\Rightarrow 11^2 + 2^2 = 10^2 + 5^2 = 125$$

Thus,725, 650,629,221,125 represent second order Ramanujan numbers with base numbers as real integers.

It is worth to mention that one may obtain second order Ramanujan numbers with base numbers as Gaussian integers.

Illustration 3.2

Consider from Table 3.1

$$y_0(k,-1,1) = 4k^2 = 4k^2 * 1 = 4k * k$$
$$= A * B \quad = C * D, \text{say}$$

From the above relation, one may observe that

$$(A+iB)^2 + (C-iD)^2 = (A-iB)^2 + (C+iD)^2 = A^2 - B^2 + C^2 - D^2$$
$$(4k^2+i)^2 + (4k-ik)^2 = (4k^2-i)^2 + (4k+ik)^2$$
$$= 16k^4 + 15k^2 - 1$$

Thus, $16k^4 + 15k^2 - 1$ represents the second order Ramanujan number with base numbers as Gaussian integers.

Special case

The substitution of the linear transformations

$$x = u+v, y = u-v, z = ku, u \neq v \neq 0, k > 0 \qquad (3.7)$$

in (3.3) leads to the pellian equation

$$Y^2 = 12v^2 + (k^2)^2 \qquad (3.8)$$

where

$$Y = 2u - k^2 \qquad (3.9)$$

The smallest positive integer solutions to (3.8) are given by

$$v_0 = 2k^2, Y_0 = 7k^2 \qquad (3.10)$$

To obtain the other solutions to (3.9), consider the pellian equation

$$Y^2 = 12v^2 + 1$$

whose general solution is given by

$$\tilde{Y}_s = \frac{1}{2}f_s, \tilde{v}_s = \frac{1}{2\sqrt{12}}g_s \qquad (3.11)$$

where

$$f_s = (7 + 2\sqrt{12})^{s+1} + (7 - 2\sqrt{12})^{s+1},$$

$$g_s = (7 + 2\sqrt{12})^{s+1} - (7 - 2\sqrt{12})^{s+1}.$$

Employing the lemma of Brahmagupta between the solutions (3.10) & (3.11), we have

$$v_{s+1} = v_0 \tilde{Y}_s + Y_0 \tilde{v}_s = k^2 f_s + \frac{7}{2\sqrt{12}}k^2 g_s = \frac{1}{4}[4k^2 f_s + \frac{14}{\sqrt{12}}k^2 g_s],$$

$$Y_{s+1} = Y_0 \tilde{Y}_s + 3v_0 \tilde{v}_s = \frac{7}{2}k^2 f_s + \sqrt{12}k^2 g_s \ , s = -1,0,1,...$$

In view of (3.9) ,we get

$$u_{s+1} = \frac{1}{2}[Y_{s+1} + k^2] = \frac{1}{4}[7k^2 f_s + 2\sqrt{12}k^2 g_s + 2k^2]$$

From (3.7), we have

$$x_{s+1}(k,s) = \frac{1}{4}[2k^2 + 11k^2 f_s + \frac{38}{\sqrt{12}}k^2 g_s],$$

$$y_{s+1}(k,s) = \frac{1}{4}[2k^2 + 3k^2 f_s + \frac{10}{\sqrt{12}}k^2 g_s],$$

$$z_{s+1}(k,s) = \frac{k^3}{4}[2 + 7f_s + 2\sqrt{12}g_s], s = -1,0,1,...$$

A few numerical solutions to (3.1) are given in Table 3.2 below:

Table 3.2-Numerical solutions

s	$x_{s+1}(k,s)$	$y_{s+1}(k,s)$	$z_{s+1}(k,s)$

-1	$6k^2$	$2k^2$	$4k^3$
0	$77k^2$	$21k^2$	$49k^3$
1	$1066k^2$	$286k^2$	$676k^3$

The recurrence relations satisfied by the solutions of (3.1) are given below:

$$x_{s+3}(k,s) - 14x_{s+2}(k,s) + x_{s+1}(k,s) = -6k^2$$
$$y_{s+3}(k,s) - 14y_{s+2}(k,s) + y_{s+1}(k,s) = -6k^2$$
$$z_{s+3}(k,s) - 14z_{s+2}(k,s) + z_{s+1}(k,s) = -6k^3$$

Interesting observations

(i) $k[x_{s+1}(k,s) + y_{s+1}(k,s)] = 2z_{s+1}(k,s)$

(ii)
$$k^3[(x_{s+1}(k,s))^3 + (y_{s+1}(k,s))^3] + 6k^2 * x_{s+1}(k,s) * y_{s+1}(k,s) * z_{s+1}(k,s) = 8\,(z_{s+1}(k,s))^3$$

(iii) $2[x_{2s+1}(k,s) + y_{2s+1}(k,s)]$ is a perfect square

(iv) $2k[x_{3s+2}(k,s) + y_{3s+2}(k,s)] + 3k[38y_{s+1}(k,s) - 10x_{s+1}(k,s)] - 44k^3 = (kf_s)^3$

(v) $38y_{2s+2}(k,s) - 10x_{2s+2}(k,s) - 12k^2 = (kf_s)^2$

(vi) $[19y_{s+1}(k,s) - 5x_{s+1}(k,s) - 7k^2]^2 - 3[3x_{s+1}(k,s) - 11y_{s+1}(k,s) + 4k^2]^2 = k^4$

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Chapter 4

TECHNIQUE TO SOLVE NON-UNIFORM DIOPHANTINE EQUATION OF DEGREE THREE WITH THREE UNKNOWNS

4.1 Technical Procedure

The non-homogeneous third degree equation is

$$2xz=(x+z)y^2 \tag{4.1}$$

The substitution of the transformations

$$x=u+v, z=u-v, u\neq v\neq 0 \tag{4.2}$$

in (4.1) gives the ternary quadratic diophantine equation

$$u^2-y^2u-v^2=0 \tag{4.3}$$

Treating (4.3) as a quadratic in u and solving for the same ,we have

$$u=\frac{y^2\pm\sqrt{y^4+4v^2}}{2} \tag{4.4}$$

To eliminate the square-root on the R.H.S. of (4.4), assume

$$\alpha^2=y^4+4v^2 \tag{4.5}$$

Choice 1

Express (4.5) as the system of double equations

$$\alpha+2v=y^4,$$
$$\alpha-2v=1.$$

which is satisfied by

$$y = 2s + 1,$$
$$\alpha = 8s^4 + 16s^3 + 12s^2 + 4s + 1, \quad (4.6)$$
$$v = 4s^4 + 8s^3 + 6s^2 + 2s.$$

From (4.4) , we have two values for u given by

$$u = 4s^4 + 8s^3 + 8s^2 + 4s + 1, - 4s^4 - 8s^3 - 4s^2$$

In view of (4.2) ,we obtain two sets of integer solutions to (4.1) as presented below :

Set 1

$$x = 8s^4 + 16s^3 + 14s^2 + 6s + 1, y = 2s + 1$$
$$z = 2s^2 + 2s + 1.$$

Set 2

$$x = 2\, s^2 + 2s, y = 2s + 1$$
$$z = -8s^4 - 16s^3 - 10s^2 - 2s.$$

Choice 2

Consider (4.5) as the pair of equations

$$\alpha + 2v = y^3,$$
$$\alpha - 2v = y. \quad (4.7)$$

which is satisfied by

$$\alpha = \frac{y^3 + y}{2}, v = \frac{y^3 - y}{2}$$

Note that there are three patterns of integer solutions to the system of double equations (4.7) and are presented below jointly with the corresponding integer solutions to (4.1).

Pattern 1

The assumption

$$y = 4s \quad (4.8)$$

gives

$$\alpha = 32s^3 + 2s, v = 16s^3 - s$$

From (4.4) , we have two values for u given by

$$u = 16s^3 + 8s^2 + s, -16s^3 + 8s^2 - s$$

In view of (4.2) ,we obtain two sets of integer solutions to (4.1) as presented below :

Set 3

$$x = 32s^3 + 8s^2 , y = 4s$$
$$z = 8s^2 + 2s.$$

Set 4

$$x = 8 s^2 - 2s, y = 4s$$
$$z = -32s^3 + 8s^2 .$$

Pattern 2

The assumption

$$y = 4s + 1 \tag{4.9}$$

gives

$$\alpha = 32s^3 + 24s^2 + 8s + 1, v = 16s^3 + 12 s^2 + 2s$$

From (4.4) , we have two values for u given by

$$u = 16s^3 + 20s^2 + 8s + 1, -16s^3 - 4s^2$$

In view of (4.2) ,we obtain two sets of integer solutions to (4.1) as presented below :

Set 5

$$x = 32s^3 + 32s^2 + 10s + 1, y = 4s + 1$$
$$z = 8s^2 + 6s + 1.$$

Set 6

$$x = 8 s^2 + 2s, y = 4s + 1$$
$$z = -32s^3 - 16s^2 - 2s.$$

Pattern 3

The assumption

$$y = 4s - 1 \tag{4.10}$$

gives

$$\alpha = 32s^3 - 24s^2 + 8s - 1, v = 16s^3 - 12\ s^2 + 2s$$

From (4.4) , we have two values for u given by

$$u = -16s^3 + 20s^2 - 8s + 1, 16s^3 - 4s^2$$

In view of (4.2) ,we obtain two sets of integer solutions to (4.1) as presented below :

Set 7

$$x = 8s^2 - 6s + 1, y = 4s - 1$$
$$z = -32\ s^3 + 32s^2 - 10s + 1.$$

Set 8

$$x = 32s^3 - 16\ s^2 + 2s, y = 4s - 1$$
$$z = 8\ s^2 - 2s.$$

Choice 3

Consider (4.5) as

$$\alpha + y^2 = 4v,$$
$$\alpha - y^2 = v.$$

The above system is satisfied by

$$\alpha = \frac{5v}{2}, y^2 = \frac{3v}{2} \tag{4.11}$$

Choosing

$$v = 6\beta^2$$

in (4.11) , we have

$$\alpha = 15\beta^2, y = \pm 3\beta$$

From (4.4) ,we get

$$u = 12\beta^2, -3\beta^2$$

In view of (4.2) ,we obtain two sets of integer solutions to (4.1) as presented below :

Set 9

$$x = 18\beta^2 , y = \pm 3\beta , z = 6\beta^2$$

Set 10

$$x = 3\beta^2 , y = \pm 3\beta , z = -9\beta^2$$

Choice 4

Consider (4.5) to be

$$\alpha + y^2 = v^2 ,$$
$$\alpha - y^2 = 4.$$

Solving the above system of equations, we have

$$\alpha = \frac{v^2 + 4}{2} , y^2 = \frac{v^2 - 4}{2} \tag{4.12}$$

Assuming

$$v = 2k \tag{4.13}$$

in (4.12) , we have

$$\alpha = 2k^2 + 2 \tag{4.14}$$

and

$$y^2 = 2k^2 - 2 \tag{4.15}$$

It is worth to mention that (4.15) represents negative pellian equation. After performing some algebra, the n[th] solution for (4.15) is

$$\begin{aligned} k_{n+1} &= \frac{3f_n}{2} + \sqrt{2}\, g_n , \\ y_{n+1} &= 2f_n + \frac{3\sqrt{2}\, g_n}{2} , n = -1,0,1,... \end{aligned} \tag{4.16}$$

where

$$f_n = (3 + 2\sqrt{2})^{n+1} + (3 - 2\sqrt{2})^{n+1} ,$$
$$g_n = (3 + 2\sqrt{2})^{n+1} - (3 - 2\sqrt{2})^{n+1} .$$

From (4.13) ,we get

$$v_{n+1} = 2k_{n+1} = 3f_n + 2\sqrt{2}\,g_n$$

From (4.4),we get

$$u_{n+1} = \frac{y^2{}_{n+1}}{2} + k^2{}_{n+1} + 1$$

From (4.2), we have

$$x_{n+1} = \frac{y_{n+1}^2}{2} + (k_{n+1} + 1)^2 ,$$

$$z_{n+1} = \frac{y_{n+1}^2}{2} + (k_{n+1} - 1)^2$$

where y_{n+1}, k_{n+1} are given by (4.16) .

A few integer solutions to (4.1) are shown below:

$$x_0 = 24, y_0 = 4, z_0 = 12$$

$$x_1 = 612, y_1 = 24, z_1 = 544$$

$$x_2 = 19800, y_2 = 140, z_2 = 19404$$

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Chapter 5

A GLIMPSE ON NON-UNIFORM INDETERMINATE THIRD DEGREE EQUATION WITH THREE PARAMETERS

5.1 Technical Procedure

The non-homogeneous ternary cubic equation under consideration is

$$a x^2 + b y^2 = (a+b) z^3 \tag{5.1}$$

By inspection, the following choices of x, y, z satisfy (5.1)

$$x = (m \mp b n) z,$$
$$y = (m \pm a n) z,$$
$$z = m^2 + a b n^2$$

and

$$x = (m \mp b n)\,(m^2 + a b n^2)\,(a+b)^3,$$
$$y = (m \pm a n)\,(m^2 + a b n^2)\,(a+b)^3,$$
$$z = (m^2 + a b n^2)\,(a+b)^2.$$

However, there are many more choices of integer solutions to (5.1). The process of obtaining other choices of integer solutions to (5.1) is as below:

Choice 1

The option

$$x = k y \tag{5.2}$$

in (5.1) gives

$$(a k^2 + b) y^2 = (a+b) z^3$$

which is satisfied by

$$y=(a k^2+b)\,(a+b)^2\,\alpha^{3s}\,,$$
$$z=(a k^2+b)(a+b)\,\alpha^{2s}\,,\alpha>1,s\geq 0 \tag{5.3}$$

From (5.2) ,one has

$$x=k(a k^2+b)\,(a+b)^2\,\alpha^{3s} \tag{5.4}$$

Thus, (5.3) and (5.4) satisfy (5.1).

Choice 2

The option

$$y=kx \tag{5.5}$$

in (5.1) gives

$$(a+bk^2)x^2=(a+b)z^3$$

whose solutions are

$$x=(a+bk^2)(a+b)^2\,\alpha^{3s}$$
$$z=(a+bk^2)\,(a+b)\,\alpha^{2s} \tag{5.6}$$

From (5.5), one has

$$y=k(a+bk^2)\,(a+b)^2\,\alpha^{3s} \tag{5.7}$$

Thus, (5.6) and (5.7) satisfy (5.1).

Choice 3

The substitution

$$x=z-bT,y=z+aT \tag{5.8}$$

in (5.1) leads to

$$abT^2=z^2(z-1)$$

which is satisfied by

$$z=1+abs^2\,,$$
$$T=s\,(1+abs^2). \tag{5.9}$$

From (5.8) ,one has

$$x=(1+a\,b\,s^{2})(1-b\,s),$$
$$y=(1+a\,b\,s^{2})\,(1+a\;s). \tag{5.10}$$

Thus, (5.1) is satisfied by (5.9) and (5.10).

Observations

(i) $a\,x+b\,y=(a+b)\,z$

(ii) $a\,s\,x-y+z^{2}=0$

(iii) $b\,s\,y+x-z^{2}=0$

Note 1

Apart from (5.8), take

$$x=z+bT,\,y=z-a\,T$$

and (5.1) is satisfied by

$$x=(1+a\,b\,s^{2})(1+b\,s),$$
$$y=(1+a\,b\,s^{2})\,(1-a\;s),$$
$$z=(1+a\,b\,s^{2}).$$

Choice 4

The option

$$x=X-b\,z,\,y=X+a\,z \tag{5.11}$$

in (1) gives

$$X^{2}=z^{2}\,(z-a\,b) \tag{5.12}$$

After performing some algebra , it is seen that the values of z ,X satisfying (5.12) are given by

$$z=z_{n}=a\,b+(s+n)^{2},$$
$$X=X_{n}=(a\,b+(s+n)^{2})(s+n) \tag{5.13}$$

From (5.11) ,it is obtained that

$$x_n = (ab + (s+n)^2)\,(s+n-b),$$
$$y_n = (ab + (s+n)^2)\,(s+n+a). \qquad (5.14)$$

Thus, (5.1) is satisfied by (5.13) and (5.14).

Observations

(i) $y_n - x_n = (a+b)\, z_n$

(ii) $(b y_n + a x_n)^2 = (z_n - ab)\,(y_n - x_n)^2$

(iii) $(x_n + b z_n)^2 = (y_n - a z_n)^2 = {z_n}^2\,(z_n - ab)$

(iv) $(y_n - x_n)^2[(x_n + b z_n)^2] = (y_n - x_n)^2[(y_n - a z_n)^2] = (b y_n + a x_n)^2\, {z_n}^2$

Note 2

In addition to (5.11) , one may also consider the substitution

$$x = X + bz, y = X - az$$

In this case,(5.1) is satisfied by

$$x_n = (ab + (s+n)^2)\,(s+n+b),$$
$$y_n = (ab + (s+n)^2)\,(s+n-a)\ ,$$
$$z_n = (ab + (s+n)^2)$$

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Chapter 6

A SKETCH OF INTEGER SOLUTIONS TO QUATERNARY UNIFORM CUBIC EQUATION

6.1 Technical Procedure

The quaternary third degree equation is

$$x^3 + y^3 + 24zw^2 = 3xy(x+y) \tag{6.1}$$

The choice

$$x = u+v, y = u-v, z = u, u \neq v \neq 0 \tag{6.2}$$

in (6.1) gives

$$u^2 = 3v^2 + 6w^2 \tag{6.3}$$

The procedure for solving (6.1) is presented below:

Pattern 1

Taking

$$v = X+6T, w = X-3T, u = 3U \tag{6.4}$$

in (6.3) ,we have

$$U^2 = X^2 + 18T^2 \tag{6.5}$$

which is satisfied by

$$T = 2rs, X = 18r^2 - s^2, U = 18r^2 + s^2 \tag{6.6}$$

Substituting (6.6) in (6.4) and employing (6.2) , the corresponding integer solutions to (6.1) are represented by

$$x = 3U + X + 6T = 72r^2 + 2s^2 + 12rs$$
$$y = 3U - X - 6T = 36r^2 + 4s^2 - 12rs$$
$$z = 3U = 54r^2 + 3s^2$$
$$w = X - 3T = 18r^2 - s^2 - 6rs$$

Note 1

Apart from (6.4) , one may also consider the transformations as

$$v = X - 6T, w = X + 3T, u = 3U$$

For this choice , the corresponding integer solutions to (6.1) are given by

$$x = 3U + X - 6T = 72r^2 + 2s^2 - 12rs$$
$$y = 3U - X + 6T = 36r^2 + 4s^2 + 12rs$$
$$z = 3U = 54r^2 + 3s^2$$
$$w = X + 3T = 18r^2 - s^2 + 6rs$$

Pattern 2

Write (6.5) as the pair of equations presented in Table 1 below:

Table 1- The pair of equations

Pair	I	II	III	IV	V	VI
$U+X$	$9T^2$	$3T^2$	T^2	$18T$	$9T$	$6T$
$U-X$	2	6	18	T	$2T$	$3T$

Consider System I. Solving the pair of equations , we have

$$U = \frac{9T^2 + 2}{2}, X = \frac{9T^2 - 2}{2}$$

For obtaining integer solutions ,take

$$T = 2s$$
$$\Rightarrow U = 18s^2 + 1, X = 18s^2 - 1$$

Thus, (6.1) is satisfied by

$$x = 3U + X + 6T = 72s^2 + 2 + 12s$$
$$y = 3U - X - 6T = 36s^2 + 4 - 12s$$
$$z = 3U = 54s^2 + 3$$
$$w = X - 3T = 18s^2 - 1 - 6s$$

Consider System II. Solving the pair of equations , we have

$$U = \frac{3T^2 + 6}{2}, X = \frac{3T^2 - 6}{2}$$

For obtaining integer solutions ,take

$$T = 2s$$
$$\Rightarrow U = 6s^2 + 3, X = 6s^2 - 3$$

Thus, (6.1) is satisfied by

$$x = 3U + X + 6T = 24s^2 + 6 + 12s$$
$$y = 3U - X - 6T = 12s^2 + 12 - 12s$$
$$z = 3U = 18s^2 + 9$$
$$w = X - 3T = 6s^2 - 3 - 6s$$

Consider System III. Solving the pair of equations , we have

$$U = \frac{T^2 + 18}{2}, X = \frac{T^2 - 18}{2}$$

For obtaining integer solutions ,take

$$T = 2s$$
$$\Rightarrow U = 2s^2 + 9, X = 2s^2 - 9$$

Thus , the integer solutions to (6.1) are given by

$$x = 3U + X + 6T = 8s^2 + 18 + 12s$$
$$y = 3U - X - 6T = 4s^2 + 36 - 12s$$
$$z = 3U = 6s^2 + 27$$
$$w = X - 3T = 2s^2 - 9 - 6s$$

Consider System IV. Solving the pair of equations , we have

$$U=\frac{19T}{2}, X=\frac{17T}{2}$$

For obtaining integer solutions ,take

$$T=2s$$
$$\Rightarrow U=19s, X=17s$$

Thus, (6.1) is satisfied by

$$x=3U+X+6T=86\,s$$
$$y=3U-X-6T=28\,s$$
$$z=3U=57s$$
$$w=X-3T=11\,s$$

Consider System V. Solving the pair of equations , we have

$$U=\frac{11T}{2}, X=\frac{7T}{2}$$

For obtaining integer solutions ,take

$$T=2s$$
$$\Rightarrow U=11s, X=7s$$

Thus, (6.1) is satisfied by

$$x=3U+X+6T=52\,s$$
$$y=3U-X-6T=14\,s$$
$$z=3U=33s$$
$$w=X-3T=s$$

Consider System VI. Solving the pair of equations , we have

$$U=\frac{9T}{2}, X=\frac{3T}{2}$$

For obtaining integer solutions ,take

$$T=2s$$
$$\Rightarrow U=9s, X=3s$$

Thus, (6.1) is satisfied by

$$x = 3U + X + 6T = 42\,s$$
$$y = 3U - X - 6T = 12\,s$$
$$z = 3U = 27s$$
$$w = X - 3T = -3\,s$$

Pattern 3

Write (6.3) as

$$u^2 - 3v^2 = 6w^2 \tag{6.7}$$

Assume

$$w = a^2 - 3b^2 \tag{6.8}$$

The integer 6 in (6.7) is written as

$$6 = (3+\sqrt{3})\,(3-\sqrt{3}) \tag{6.9}$$

Substituting (6.8) & (6.9) in (6.7) and employing factorization, consider

$$u + \sqrt{3}\,v = (3+\sqrt{3})\,(a+\sqrt{3}\,b)^2 \tag{6.10}$$

On comparing the terms in (6.10), one has

$$u = 3\,f(a,b) + 3g(a,b),$$
$$v = f(a,b) + 3g(a,b)$$

in which

$$f(a,b) = a^2 + 3b^2\,, g(a,b) = 2ab$$

From (6.2) ,(6.1) is satisfied by

$$x = 4f(a,b) + 6g(a,b),$$
$$y = 2f(a,b),$$
$$z = 3[f(a,b) + g(a,b)]$$

jointly with (6.8) .

Pattern 4

Write (6.7) as

$$u^2 - 3v^2 = 6\,w^2 * 1 \tag{6.11}$$

Write integer 1 in (6.11) as

$$1=\frac{[3r^2+s^2+\sqrt{3}(2rs)]\,[3r^2+s^2-\sqrt{3}(2rs)]}{(3r^2-s^2)^2} \qquad (6.12)$$

Assume

$$w=(3r^2-s^2)^2(a^2-3b^2) \qquad (6.13)$$

Substituting (6.9) ,(6.12) & (6.13) in (6.11) and employing factorization , consider

$$u+\sqrt{3}v=(3+\sqrt{3})(3r^2-s^2)^2(a+\sqrt{3}b)^2\frac{[3r^2+s^2+\sqrt{3}(2rs)]}{(3r^2-s^2)}$$
$$=(3r^2-s^2)\,(3+\sqrt{3})\,[f(a,b)+\sqrt{3}g(a,b)]\,[F(r,s)+\sqrt{3}G(r,s)]$$
$$=(3r^2-s^2)\{[3f(a,b)+3g(a,b)]+\sqrt{3}[f(a,b)+3g(a,b)]\}[F(r,s)+\sqrt{3}G(r,s)]$$

(6.14)

where

$$F(r,s)=3r^2+s^2\,,G(r,s)=2rs$$

On comparing the terms in (6.14), one has

$$u=(3r^2-s^2)\{F(r,s)[3f(a,b)+3g(a,b)]+3G(r,s)[f(a,b)+3g(a,b)]\}$$
$$v=(3r^2-s^2)\{F(r,s)[f(a,b)+3g(a,b)]+G(r,s)[3f(a,b)+3g(a,b)]\}$$

From (6.2) ,(6.1) is satisfied by

$$x=(3r^2-s^2)\{F(r,s)[4f(a,b)+6g(a,b)]+G(r,s)[6f(a,b)+12g(a,b)]\}\,,$$
$$y=(3r^2-s^2)\{F(r,s)[2f(a,b)]+G(r,s)[6g(a,b)]\},$$
$$z=(3r^2-s^2)\{F(r,s)[3f(a,b)+3g(a,b)]+G(r,s)[3f(a,b)+9g(a,b)]\}$$

jointly with (6.13) .

Pattern 5

Write (6.3) as

$$u^2-6w^2=3v^2 \qquad (6.15)$$

Assume

$$v = a^2 - 6b^2 \tag{6.16}$$

Write integer 3 in (6.15) as

$$3 = (3+\sqrt{6})\,(3-\sqrt{6}) \tag{6.17}$$

Substituting (6.16) & (6.17) in (6.15) and employing factorization, consider

$$u + \sqrt{6}\,w = (3+\sqrt{6})\,(a+\sqrt{6}\,b)^2 \tag{6.18}$$

On comparing the terms in (6.18), one has

$$\begin{aligned} u &= 3\,f(a,b) + 6\,g(a,b), \\ w &= f(a,b) + 3g(a,b) \end{aligned} \tag{6.19}$$

where

$$f(a,b) = a^2 + 6b^2,\ g(a,b) = 2ab$$

From (6.2),(6.1) is satisfied by

$$\begin{aligned} x &= 4f(a,b) - 12b^2 + 6g(a,b), \\ y &= 2f(a,b) + 12b^2 + 6g(a,b), \\ z &= 3[f(a,b) + 2\,g(a,b)] \end{aligned}$$

jointly with w in (6.19) .

Pattern 6

Write (6.15) as

$$u^2 - 6w^2 = 3\,v^2 * 1 \tag{6.20}$$

Write integer 1 in (6.20) as

$$1 = \frac{[6r^2 + s^2 + \sqrt{6}\,(2rs)]\,[6r^2 + s^2 - \sqrt{6}\,(2rs)]}{(6r^2 - s^2)^2} \tag{6.21}$$

Assume

$$v = (6r^2 - s^2)^2\,(a^2 - 6b^2) \tag{6.22}$$

Substituting (6.17), (6.21) & (6.22) in (6.20) and employing factorization, consider

$$u+\sqrt{6}\,w=(3+\sqrt{6})(6r^2-s^2)^2(a+\sqrt{6}\,b)^2\frac{[6r^2+s^2+\sqrt{6}(2rs)]}{(6r^2-s^2)}$$
$$=(6r^2-s^2)\,(3+\sqrt{6})\,[f(a,b)+\sqrt{6}\,g(a,b)]\,[F(r,s)+\sqrt{6}\,G(r,s)]$$
$$=(6r^2-s^2)\{[3f(a,b)+6g(a,b)]+\sqrt{6}[f(a,b)+3g(a,b)]\}[F(r,s)+\sqrt{6}\,G(r,s)]$$

(6.23)

where

$$F(r,s)=6r^2+s^2\,,G(r,s)=2rs$$

On comparing the terms in (6.23), one has

$$u=(6r^2-s^2)\{F(r,s)[3f(a,b)+6g(a,b)]+6G(r,s)[f(a,b)+3g(a,b)]\}$$
$$w=(6r^2-s^2)\{F(r,s)[\,f(a,b)+3g(a,b)]+G(r,s)[3f(a,b)+6g(a,b)]\}$$ (6.24)

From (6.2), (6.1) is satisfied by

$$x=(6r^2-s^2)\{F(r,s)[\,3f(a,b)+6\,g(a,b)]+G(r,s)[6\,f(a,b)+18\,g(a,b)]\}$$
$$+(6r^2-s^2)^2\,(a^2-6b^2)\;,$$
$$y=(6r^2-s^2)\{F(r,s)[\,3f(a,b)+6\,g(a,b)]+G(r,s)[6\,f(a,b)+18\,g(a,b)]\}$$
$$-(6r^2-s^2)^2\,(a^2-6b^2),$$
$$z=(6r^2-s^2)\{F(r,s)[\,3f(a,b)+6\,g(a,b)]+G(r,s)[6\,f(a,b)+18\,g(a,b)]\}$$

jointly with w in (6.24) .

Pattern 7

The solutions to (6.3) are

$$v=v_0=w,u=u_0=3w$$

Consider the pellian equation

$$u^2=3v^2+1$$

whose general solution $(\tilde{v}_n\,,\tilde{u}_n)$ is given by

$$\tilde{v}_n = \frac{1}{2\sqrt{3}} g_n \ , \tilde{u}_n = \frac{1}{2} f_n$$

where

$$f_n = (2+\sqrt{3})^{n+1} + (2-\sqrt{3})^{n+1},$$
$$g_n = (2+\sqrt{3})^{n+1} - (2-\sqrt{3})^{n+1}$$

Employing the lemma of Brahmagupta between (v_0, u_0) and $(\tilde{v}_n, \tilde{u}_n)$,we have

$$v_{n+1} = v_0 \tilde{u}_n + u_0 \tilde{v}_n = \frac{w}{2}[f_n + \sqrt{3} g_n],$$
$$u_{n+1} = u_0 \tilde{u}_n + 3 v_0 \tilde{v}_n = \frac{w}{2}[3 f_n + \sqrt{3} g_n]$$

In view of (6.2) ,the integer solutions to (6.1) are given by

$$x_{n+1} = u_{n+1} + v_{n+1} = w[2 f_n + \sqrt{3} g_n],$$
$$y_{n+1} = u_{n+1} - v_{n+1} = w\ f_n,$$
$$z_{n+1} = u_{n+1} = \frac{w}{2}[3 f_n + \sqrt{3} g_n],$$

where w is chosen arbitrarily.

A few numerical solutions to (6.1) are presented below:

$$x_0 = 4w, y_0 = 2w, z_0 = 3w$$
$$x_1 = 14w, y_1 = 4w, z_1 = 9w$$
$$x_2 = 52w, y_2 = 14w, z_2 = 33w$$
$$x_3 = 194w, y_3 = 52w, z_3 = 123w \ .$$

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Chapter 7

ON CUBIC EQUATION WITH FOUR UNKNOWNS

7.1 Technical Procedure

The cubic Diophantine equation with four unknowns studied for its non-zero distinct integer solutions is given by

$$x^3+y^3+(x+y)(x-y)^2=16\,z\,w^2 \tag{7.1}$$

Introduction of the linear transformations

$$x=u+v,\; y=u-v,\; z=u,\; u\neq v\neq 0 \tag{7.2}$$

in (7.1) leads to

$$u^2+7v^2=8w^2 \tag{7.3}$$

Now, we solve (7.3) through different methods and thus obtain different patterns of solutions to (7.1).

Method I:

Assume

$$w=a^2+7b^2 \tag{7.4}$$

where a and b are non-zero distinct integers.

Write integer 8 in (7.3) as

$$8=\left(1+i\sqrt{7}\right)\left(1-i\sqrt{7}\right) \tag{7.5}$$

Using (7.4) and (7.5) in (7.3) and applying the method of factorization, it is written as the system of double equations as

$$u+i\sqrt{7}\,v=\left(1+i\sqrt{7}\right)\left(a+i\sqrt{7}b\right)^2=\left(1+i\sqrt{7}\right)[f(a,b)+i\sqrt{7}\,g(a,b)]$$
$$u-i\sqrt{7}v=\left(1-i\sqrt{7}\right)\left(a-i\sqrt{7}b\right)^2=\left(1-i\sqrt{7}\right)[f(a,b)-i\sqrt{7}\,g(a,b)]$$

where

$$f(a,b)=(a^2-7b^2)\ ,g(a,b)=2ab \qquad (7.6)$$

Equating the real and imaginary parts in either of the above two equations , we have

$$u=f(a,b)-7g(a,b)=a^2-7b^2-14ab$$
$$v=f(a,b)+g(a,b)=a^2-7b^2+2ab$$

From (7.2), (7.1) is satisfied by

$$x=2f(a,b)-6g(a,b)=2a^2-14b^2-12ab\ ,$$
$$y=-8g(a,b)=-16ab\ ,$$
$$z=f(a,b)-7g(a,b)=a^2-7b^2-14ab$$

jointly with (7.4) .

Note 1

The integer 8 ,apart from (7.5) , may be factorized as

$$8=\frac{(5+i\sqrt{7})(5-i\sqrt{7})}{4}$$
$$8=\frac{(11+i\sqrt{7})(11-i\sqrt{7})}{16}$$
$$8=\frac{(31+i\sqrt{7})(31-i\sqrt{7})}{121}$$

Following the above procedure , three more sets of integer solutions to (7.1) are obtained.

Method 2

Consider (7.3) as

$$u^2+7v^2=8w^2*1 \qquad (7.7)$$

Write integer 1 in (7.7) as

$$1=\frac{(a(s)+i\,b(s)\sqrt{7})\ (a(s)-i\,b(s)\sqrt{7})}{(a(s)+1)^2} \qquad (7.8)$$

where

$$a(s)=(14s^2-14s+3),\, b(s)=(2s-1) \qquad (7.9)$$

Substituting (7.4) ,(7.5) & (7.8) in (7.7) and employing factorization , we consider

$$(u+i\sqrt{3}v)=\left(1+i\sqrt{7}\right)\left(a+i\sqrt{7}b\right)^2\frac{[a(s)+i\, b(s)\sqrt{7}]}{(a(s)+1)}$$

$$=\left(1+i\sqrt{7}\right)[f(a,b)+i\sqrt{7}g(a,b)]\,\frac{[a(s)+i\, b(s)\sqrt{7}]}{(a(s)+1)}$$

$$=\{[f(a,b)-7g(a,b)]+i\sqrt{7}[f(a,b)+g(a,b)]\}\frac{[a(s)+i\, b(s)\sqrt{7}]}{(a(s)+1)}$$

Equating the real and imaginary parts in the above equation , we have

$$\begin{aligned} u&=\frac{1}{(a(s)+1)}\{a(s)[f(a,b)-7g(a,b)]-7b(s)[f(a,b)+g(a,b)]\}\\ v&=\frac{1}{(a(s)+1)}\{b(s)[f(a,b)-7g(a,b)]+a(s)[f(a,b)+g(a,b)]\}\end{aligned} \qquad (7.10)$$

Since , the focus is on finding integer solutions , taking

$$a=(a(s)+1)A,\, b=(a(s)+1)B$$

in (7.4) & (7.10) and utilizing (7.2) ,(7.1) is satisfied by

$$\begin{aligned} x&=(a(s)+1)\{[a(s)+b(s)][f(A,B)-7g(A,B)]+[a(s)-7b(s)][f(A,B)+g(A,B)]\},\\ y&=(a(s)+1)\{[a(s)-b(s)][f(A,B)-7g(A,B)]-[a(s)+7b(s)][f(A,B)+g(A,B)]\},\\ z&=(a(s)+1)\{a(s)[f(A,B)-7g(A,B)]-7b(s)[f(A,B)+g(A,B)]\}\\ w&=(a(s)+1)^2[A^2+7B^2].\end{aligned}$$

(7.11)

To analyse the nature of solutions,one has to take particular values to s. For simplicity and clear understanding , the option s=1 gives

$$a(s) = a(1) = 3, b(s) = b(1) = 1$$

$$1 = \frac{(3 + i\sqrt{7})(3 - i\sqrt{7})}{16}$$

Also ,

$$f(A, B) = A^2 - 7B^2, g(A, B) = 2AB$$

From (7.11) ,the integer solutions to (7.1) are given by

$$x = x(A, B) = -256AB$$
$$y = y(A, B) = -32A^2 + 224B^2 - 192AB$$
$$z = z(A, B) = -16A^2 + 112B^2 - 224AB$$
$$w = w(A, B) = 16A^2 + 112B^2$$

Note 2

Apart from (7.8), one may have other representations to integer 1 which are exhibited below

Representation 1:

$$1 = \frac{(a(s) + i\, b(s)\sqrt{7})(a(s) - i\, b(s)\sqrt{7})}{(a(s) + 2)^2}$$

where

$$a(s) = (7s^2 - 1), b(s) = 2s \qquad (7.12)$$

Representation 2:

$$1 = \frac{(a(s) + i\, b(s)\sqrt{7})(a(s) - i\, b(s)\sqrt{7})}{(a(s) + 14s^2)^2}$$

where

$$a(s) = (r^2 - 7s^2), b(s) = 2rs \qquad (7.13)$$

Representation 3:

$$1=\frac{(a(s)+i\,b(s)\sqrt{7})(a(s)-i\,b(s)\sqrt{7})}{(a(s)+2s^2)^2}$$

where

$$a(s)=(7r^2-s^2),b(s)=2rs \tag{7.14}$$

Representation 4:

$$1=\frac{(1+i\sqrt{7}\alpha_n)(1-i\sqrt{7}\alpha_n)}{(\beta_n)^2}=\frac{(2+i\ g_n)(2-i\ g_n)}{(f_n)^2}$$

where

$$\alpha_n=\frac{1}{2\sqrt{7}}g_n=\frac{1}{2\sqrt{7}}[(8+3\sqrt{7})^{n+1}-(8-3\sqrt{7})^{n+1}],$$
$$\beta_n=\frac{1}{2}f_n=\frac{1}{2}[(8+3\sqrt{7})^{n+1}+(8-3\sqrt{7})^{n+1}],n=0,1,2,... \tag{7.15}$$

Using (7.12) ,(7.13) ,(7.14) and (7.15) in (7.11) in turn ,the corresponding integer solutions to (7.1) are obtained.

Method 3

Consider (7.3) as

$$8w^2-7v^2=u^2*1 \tag{7.16}$$

Assume

$$u=8a^2-7b^2 \tag{7.17}$$

Write the integer 1 in (7.16) as

$$1=(\sqrt{8}+\sqrt{7})(\sqrt{8}-\sqrt{7}) \tag{7.18}$$

Substituting (7.17) & (7.18) in (7.16) and applying factorization , consider

$$\sqrt{8}w+\sqrt{7}v=(\sqrt{8}+\sqrt{7})(\sqrt{8}a+\sqrt{7}b)^2$$

from which , on equating the corresponding terms ,one obtains

$$w = 8a^2 + 7b^2 + 14ab, \quad v = 8a^2 + 7b^2 + 16ab \tag{7.19}$$

From (7.2), (7.1) is satisfied by

$$x = 16a^2 + 16ab,$$
$$y = -14b^2 - 16ab,$$
$$z = 8a^2 - 7b^2$$

jointly with w in (7.19).

Note 3

In addition to (7.18) , we have

$$1 = \frac{(2\sqrt{8}+\sqrt{7})\,(2\sqrt{8}-\sqrt{7})}{25}$$
$$1 = \frac{(4\sqrt{8}+\sqrt{7})\,(4\sqrt{8}-\sqrt{7})}{121}$$

The repetition of the above process leads to different sets of solutions to (7.1) .

Method 4

Rewrite (7.3) as

$$8w^2 - u^2 = 7v^2 \tag{7.20}$$

Assume

$$v = 8a^2 - b^2 \tag{7.21}$$

Write the integer 7 in (7.20) as

$$7 = (\sqrt{8}+1)(\sqrt{8}-1) \tag{7.22}$$

Inserting (7.21) & (7.22) in (7.20) and using factorization ,we consider

$$\sqrt{8}\,w + u = (\sqrt{8}+1)(\sqrt{8}\,a + b)^2$$

On comparing , we have

$$w = (8a^2 + b^2) + 2ab, \quad u = (8a^2 + b^2) + 16ab \tag{7.23}$$

From (7.2), (7.1) is satisfied by

$$x = 16a^2 + 16ab,$$
$$y = 2b^2 + 16ab,$$
$$z = (8a^2 + b^2) + 16ab$$

jointly with w in (7.23) .

Note 4

Apart from (7.22) , we have

$$7 = (2\sqrt{8} + 5)(2\sqrt{8} - 5),$$
$$7 = (4\sqrt{8} + 11)(4\sqrt{8} - 11),$$
$$7 = (11\sqrt{8} + 31)(11\sqrt{8} - 31).$$

The repetition of the above process leads to different sets of solutions to (7.1) .

Method 5

Consider (7.20) as

$$8w^2 - u^2 = 7v^2 * 1 \qquad (7.24)$$

Write the integer 1 in (7.24) as

$$1 = \frac{(\sqrt{8} + 2)(\sqrt{8} - 2)}{4} \qquad (7.25)$$

Assume

$$v = 8a^2 - 4b^2 \qquad (7.26)$$

Inserting (7.26) ,(7.22) & (7.25) in (7.24) and applying factorization , we consider

$$\sqrt{8}w + u = (\sqrt{8} + 1)(\sqrt{8}a + 2b)^2 \frac{(\sqrt{8} + 2)}{2}$$

On comparing ,we have

$$w = 3(4a^2 + 2b^2) + 20ab,$$
$$u = 10(4a^2 + 2b^2) + 48ab. \qquad (7.26)$$

From (7.2), (7.1) is satisfied by

$$x = 48a^2 + 16b^2 + 48ab,$$
$$y = 32a^2 + 24b^2 + 48ab,$$
$$z = 40a^2 + 20b^2 + 48ab$$

jointly with w in (7.26).

Note 5

In addition to (7.25) ,we have

$$1 = \frac{(5\sqrt{8}+2)(5\sqrt{8}-2)}{196}$$
$$1 = \frac{(5\sqrt{8}+14)(5\sqrt{8}-14)}{4}$$

The repetition of the above process leads to different sets of solutions to (7.1) .

Method 6

Express (7.3) in the form of ratio as below:

$$\frac{u+w}{w+v} = \frac{7(w-v)}{u-w} = \frac{\alpha}{\beta}, \beta \neq 0 \tag{7.27}$$

The above equation is written as the system of double equations

$$\beta u - \alpha v + (\beta - \alpha) w = 0$$
$$\alpha u + 7\beta v - (7\beta + \alpha) w = 0$$

Employing the method of cross-multiplication , we have

$$u = \alpha^2 - 7\beta^2 + 14\alpha\beta,$$
$$v = -\alpha^2 + 7\beta^2 + 2\alpha\beta, \tag{7.28}$$
$$w = \alpha^2 + 7\beta^2$$

From (7.2), (7.1) is satisfied by

$$x = 16\alpha\beta,$$
$$y = 2\alpha^2 - 14\beta^2 + 12\alpha\beta,$$
$$z = \alpha^2 - 7\beta^2 + 14\alpha\beta$$

jointly with w given in (7.28).

Note 6

Apart from (7.27) , we may have the following ratio form representations

$$\frac{u+w}{7(w+v)} = \frac{(w-v)}{u-w} = \frac{\alpha}{\beta}, \beta \neq 0$$
$$\frac{u+w}{w-v} = \frac{7(w+v)}{u-w} = \frac{\alpha}{\beta}, \beta \neq 0$$
$$\frac{u+w}{7(w-v)} = \frac{(w+v)}{u-w} = \frac{\alpha}{\beta}, \beta \neq 0$$

Proceeding as above, we obtain three additional patterns to (7.1).

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Chapter 8

ON UNIFORM THIRD DEGREE EQUATION WITH FOUR PARAMETERS

8.1 Technical Procedure

The uniform quaternary third degree equation under consideration is

$$2(x^3+y^3)=(z+w)^2(z-w) \quad (8.1)$$

On examination, observe that the set of four integers represented by $(x,y,z,w)=(4s,2s,5s,s)$,$(-2s,-4s,-5s,-s)$. satisfies (8.1) .In addition ,there are plenty of varieties of solutions in integers for (8.1) & the procedure to determine them is analysed as follows .

Procedure 1

The insertion of the below mentioned linear transformations

$$x=u+v, y=u-v, z=u+p, w=u-p \ , u\neq v, p \quad (8.2)$$

to (8.1) reduces it to uniform second degree equation having three variables

$$(u-p)^2+3v^2=p^2 \quad (8.3)$$

Assume

$$p=a^2+3b^2 \quad (8.4)$$

Substituting (8.4) in (8.3) & applying factorizing technique , define

$$(u-p)+i\sqrt{3}v=(a+i\sqrt{3}b)^2 \quad (8.5)$$

On comparing the corresponding terms of (8.5) and simplifying, one has

$$u=2a^2, v=2ab. \quad (8.6)$$

In view of (8.2) ,from (8.4) & (8.6) ,one obtains the integer solutions to (8.1) to be

$$x = 2a^2 + 2ab,$$
$$y = 2a^2 - 2ab, \qquad (8.7)$$
$$z = 3(a^2 + b^2),$$
$$w = (a^2 - 3b^2).$$

Procedure 2

Write (8.3) as

$$(u - p)^2 + 3v^2 = p^2 * 1$$

In (8.8) ,consider 1 to be (8.8)

$$1 = \frac{(1 + i\sqrt{3})\,(1 - i\sqrt{3})}{4} \qquad (8.9)$$

Inserting (8.4) & (8.9) in (8.8) & implementing factorizing technique , write

$$u - p + i\sqrt{3}v = \frac{(a + i\sqrt{3}b)^2 (1 + i\sqrt{3})}{2} \qquad (8.10)$$

In (8.10) ,on comparing the respective terms ,it is seen that

$$u = \frac{3(a^2 + b^2)}{2} - 3ab,$$
$$v = \frac{(a^2 - 3b^2)}{2} + ab. \qquad (8.11)$$

In view of (8.2) , from (8.4) & (8.11) ,one obtains the integer solutions to (8.1) to be

$$x = 2a^2 - 2ab,$$
$$y = a^2 + 3b^2 - 4ab,$$
$$z = \frac{5a^2 + 9b^2 - 6ab}{2}, \qquad (8.12)$$
$$w = \frac{a^2 - 3b^2 - 6ab}{2}.$$

where a, b are of the same parity.

Note 1

Also , in (8.8), represent 1 in the following forms :

$$\mathbf{1=\frac{(3a^2-b^2+i\sqrt{3}\,2ab)\,(3a^2-b^2-i\sqrt{3}\,2ab)}{(3a^2+b^2)^2}},$$

$$\mathbf{1=\frac{(a^2-3b^2+i\sqrt{3}\,2ab)\,(a^2-3b^2-i\sqrt{3}\,2ab)}{(a^2+3b^2)^2}}$$

$$1=\frac{(a(s)+i\;b(s)\sqrt{3})(a(s)-i\,b(s)\sqrt{3})}{(a(s)+1)^2}$$

where

$$a(s)=(6s^2-6s+1),b(s)=(2s-1)$$

$$1=\frac{(a(s)+i\;b(s)\sqrt{3})(a(s)-i\,b(s)\sqrt{3})}{(a(s)+2)^2}$$

where

$$a(s)=(3s^2-1),b(s)=2s$$

$$1=\frac{(1+i\;\sqrt{3}\alpha_n)(1-i\;\sqrt{3}\alpha_n)}{(\beta_n)^2}=\frac{(2+i\;g_n)(2-i\;g_n)}{(f_n)^2}$$

where

$$\alpha_n=\frac{1}{2\sqrt{3}}g_n=\frac{1}{2\sqrt{3}}[(2+\sqrt{3})^{n+1}-(2-\sqrt{3})^{n+1}],$$

$$\beta_n=\frac{1}{2}f_n=\frac{1}{2}[(2+\sqrt{3})^{n+1}+(2-\sqrt{3})^{n+1}],n=0,1,2,...$$

Adopting a similar analysis, various choices of solutions in integers for (8.1) are determined.

Procedure 3

Rewrite (8.3) as

$$p^2-3v^2=(u-p)^2*1 \qquad (8.13)$$

Take

$$u-p=a^2-3b^2 \qquad (8.14)$$

Assume the integer 1 on the R.H.S. of (8.13) as

$$1=(2+\sqrt{3})(2-\sqrt{3}) \tag{8.15}$$

Substituting (8.14) & (8.15) in (8.13) and applying factorization, we consider

$$p+\sqrt{3}v=(2+\sqrt{3})\,(a+\sqrt{3}b)^2$$

On comparing the respective terms in the above equation ,it is seen that

$$v=a^2+3b^2+4ab \tag{8.16}$$

and

$$p=2(a^2+3b^2)+6ab \tag{8.17}$$

From (8.14) & (8.17), we get

$$u=3(a^2+b^2)+6ab \tag{8.18}$$

Substituting the above values of u, v, p from (8.18), (8.16) & (8.17) in (8.2), the integer solutions satisfying (8.1) are given by

$$\begin{aligned} x &= 4a^2+6b^2+10ab, \\ y &= 2a^2+2ab, \\ z &= 5a^2+9b^2)+12ab, \\ w &= a^2-3b^2. \end{aligned} \tag{8.19}$$

Note 2

In addition to (8.15), the integer 1 is written as

$$\mathbf{1=\frac{(p^2+3q^2+\sqrt{3}\,2pq)\,(p^2-3q^2-\sqrt{3}\,2pq)}{(p^2-3q^2)^2}},$$

$$\mathbf{1=\frac{(3p^2+q^2+\sqrt{3}\,2pq)\,(3p^2+q^2-\sqrt{3}\,2pq)}{(3p^2-q^2)^2}}$$

$$1=\frac{(a(s)+1+b(s)\sqrt{3})(a(s)+1-b(s)\sqrt{3})}{(a(s))^2}$$

where

$$a(s)=(6s^2-6s+1), b(s)=(2s-1)$$

$$1=\frac{(a(s)+2+b(s)\sqrt{3})(a(s)+2-b(s)\sqrt{3})}{(a(s))^2}$$

where

$$a(s)=(3s^2-1),b(s)=2s$$

Following the above procedure, different patterns of integer solutions to (8.1) are obtained.

Process 4

Write (8.3) as the pair of equations as below

$$\begin{aligned} p+(u-p)&=v^2, \\ p-(u-p)&=3. \end{aligned} \tag{8.20}$$

Solving the above system of double equations , one obtains

$$u=v^2=2p-3 \tag{8.21}$$

By scrutiny, from (8.21)

$$v=2k+1,u=4k^2+4k+1,p=2k^2+2k+2$$

From (8.2),

$$\begin{aligned} x&=4k^2+6k+2, \\ y&=4k^2+2k, \\ z&=6k^2+6k+3, \\ w&=2k^2+2k-1. \end{aligned}$$

satisfy (8.1).

Note 3

It is worth to mention that (8.3) is also written as the pair of equations as below :

$$\begin{aligned} p+(u-p)&=3v^2, \\ p-(u-p)&=1. \end{aligned}$$

Solving the above system of double equations , one obtains

$$u=3v^2=2p-3 \tag{8.22}$$

By scrutiny, from (8.22)

$$v = 2k + 1, u = 12k^2 + 12k + 3, p = 6k^2 + 6k + 2$$

From (8.2),

$$x = 12k^2 + 14k + 4,$$
$$y = 12k^2 + 10k + 2,$$
$$z = 18k^2 + 18k + 5,$$
$$w = 6k^2 + 6k - 1.$$

satisfy (8.1).

Remark

The assumption

$$x = 2u + 2v,\ \ y = 2u - 2v,\ \ z = 2u + p,\ \ w = 2u - p$$

in (8.1) gives

$$(2u - p)^2 + 12v^2 = p^2$$

Following a similar analysis as above, other patterns of integer solutions to (8.1) are obtained.

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Chapter 9

ON QUATERNARY EQUAL THIRD DEGREE EQUATION

9.1 Technical Procedure

The uniform third degree equation having four variables is

$$x^3 + y^3 = 62zw^2 \tag{9.1}$$

The substitution of the transformations

$$x = u + v, y = u - v, z = u, u \neq \pm v \neq 0 \tag{9.2}$$

in (9.1) gives

$$u^2 + 3v^2 = 31w^2 \tag{9.3}$$

The procedure to obtain patterns of integer solutions to (9.1) is illustrated:

Pattern 1

The option

$$v = 3k, k \neq 0 \tag{9.4}$$

in (9.3) leads to negative pell equation

$$u^2 = 31w^2 - 27k^2 \tag{9.5}$$

which is satisfied by

$$w_0 = k, u_0 = 2k$$

To obtain the other solutions to (9.5), consider the corresponding pellian equation given by

$$u^2 = 31w^2 + 1$$

whose general solution $(\tilde{u}_n, \tilde{w}_n)$ is given by

$$\tilde{u}_n = \frac{1}{2} f_n , \tilde{w}_n = \frac{1}{2\sqrt{3}} g_n$$

where

$$f_n = (1520 + 273\sqrt{31})^{n+1} + (1520 - 273\sqrt{31})^{n+1} ,$$
$$g_n = (1520 + 273\sqrt{31})^{n+1} - (1520 - 273\sqrt{31})^{n+1} .$$

Employing the lemma of Brahmagupta between the solutions (w_0, u_0) and $(\tilde{w}_n, \tilde{u}_n)$, we have

$$w_{n+1} = \frac{k}{62}(31 f_n + 2\sqrt{31} g_n) \qquad (9.6)$$

and

$$u_{n+1} = \frac{k}{2}(2 f_n + \sqrt{31} g_n) \qquad (9.7)$$

From (9.4), we have

$$v_{n+1} = 3k \qquad (9.8)$$

Substituting (9.7) & (9.8) in (9.2), one obtains

$$x_{n+1} = \frac{k}{2}(2 f_n + \sqrt{31} g_n) + 3k ,$$
$$y_{n+1} = \frac{k}{2}(2 f_n + \sqrt{31} g_n) - 3k , \qquad (9.9)$$
$$z_{n+1} = \frac{k}{2}(2 f_n + \sqrt{31} g_n) .$$

Thus, (9.6) & (9.9) satisfy (9.1). In the above equations, $n = -1, 0, 1, ...$

The recurrence relations satisfied by the solutions to (9.1) are given by

$$x_{n+3} - 3040\, x_{n+2} + x_{n+1} = -9114k ,$$
$$y_{n+3} - 3040\, y_{n+2} + y_{n+1} = 9114k ,$$
$$z_{n+3} - 3040\, z_{n+2} + z_{n+1} = 0 ,$$
$$x_{n+3} - 3040\, x_{n+2} + x_{n+1} = 0 .$$

A few numerical solutions to (9.1) are given below:

$$x_0 = 5k, y_0 = -k, z_0 = 2k, w_0 = k$$
$$x_1 = 11506\,k, y_1 = 11500k, z_1 = 11503k, w_1 = 2066k$$
$$x_2 = 34969121k, y_2 = 34969115k, z_2 = 34969118k, w_2 = 6280639k$$

Pattern 2

Write (9.3) as

$$31w^2 - 3v^2 = u^2 * 1 \quad (9.10)$$

Assume

$$u = 31a^2 - 3b^2 \quad (9.11)$$

Express the integer 1 on the R.H.S. of (9.10) as the product of irrational conjugates as shown below:

$$1 = \frac{(\sqrt{31} + 3\sqrt{3})\,(\sqrt{31} - 3\sqrt{3})}{4} \quad (9.12)$$

Substituting (9.11) & (9.12) in (9.10) and employing factorization, consider

$$\sqrt{31}\,w + \sqrt{3}\,v = \frac{(\sqrt{31} + 3\sqrt{3})}{2}(\sqrt{31}\,a + \sqrt{3}\,b)^2$$

On comparing, one obtains

we get

$$w = \frac{(31a^2 + 3b^2)}{2} + 9ab\,,$$
$$v = \frac{(93a^2 + 9b^2)}{2} + 31ab. \quad (9.13)$$

Replacing a by $2A$ and b by $2B$

in (9.11) & (9.13) ,we have

$$u = 4(31A^2 - 3B^2)\,,$$
$$v = 186A^2 + 18B^2 + 124AB \quad (9.14)$$

and

$$w = 62A^2 + 6B^2 + 36AB \tag{9.15}$$

From (9.14) & (9.2), we get

$$\begin{aligned} x &= 310A^2 + 6B^2 + 124AB, \\ y &= -62A^2 - 30B^2 - 124AB, \\ z &= 124A^2 - 12B^2. \end{aligned} \tag{9.16}$$

Thus, (9.15) and (9.16) satisfy (9.1).

Pattern 3

Consider (9.3) as

$$u^2 + 3v^2 = 31w^2 * 1 \tag{9.17}$$

Assume

$$w = a^2 + 3b^2 \tag{9.18}$$

Write integers 31 & 1 in (9.17) as

$$\begin{aligned} 31 &= (2 + i3\sqrt{3})\,(2 - i3\sqrt{3}) \\ 1 &= \frac{(1 + i4\sqrt{3})\,(1 - i4\sqrt{3})}{49} \end{aligned} \tag{9.19}$$

Substituting (9.18) & (9.19) in (9.17) and employing factorization, consider

$$u + i\sqrt{3}\,v = (2 + i3\sqrt{3})\frac{(1 + i4\sqrt{3})}{7}(a + i\sqrt{3}\,b)^2$$

Proceding as in Pattern 2, (9.1) is satisfied by

$$\begin{aligned} x &= -7[23(a^2 - 3b^2) + 134ab], \\ y &= -7[45(a^2 - 3b^2) - 2ab], \\ z &= -7[34(a^2 - 3b^2) + 66ab], \\ w &= 49\,(a^2 + 3b^2). \end{aligned}$$

Note

Apart from (9.19) ,express the integers 31 & 1 on the R.H.S. of (9.17) as exhibited below:

$$31 = \frac{(7 + i45\sqrt{3})\,(7 - i5\sqrt{3})}{4}$$

$$31 = \frac{(11 + i\,\sqrt{3})\,(11 - i\sqrt{3})}{4}$$

$$1 = \frac{(1 + i\,\sqrt{3})\,(1 - i\,\sqrt{3})}{4}\;, \tag{9.20}$$

$$1 = \frac{(3a^2 - b^2 + i\,\sqrt{3}\,(2ab))\,(3a^2 - b^2 - i\,\sqrt{3}\,(2ab))}{(3a^2 + b^2)^2}\;,$$

$$1 = \frac{(a^2 - 3b^2 + i\,\sqrt{3}\,(2ab))\,(a^2 - 3b^2 - i\,\sqrt{3}\,(2ab))}{(a^2 + 3b^2)^2}\,.$$

By considering combinations between the integers 31 & 1 in (9.20) and following the process as in Pattern 3 , some more integer solutions to (9.1) are obtained.

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Chapter 10

A SKETCH ON CUBIC EQUATION WITH EQUAL TERMS AND FOUR VARIABLES

10.1 Technical Procedure

Third Degree Equation with Equal Terms and Four Variabless under consideration is

$$x^3 + y^3 = 7(z-w)^2(z+w) \tag{10.1}$$

By inspection, the lattice points given by $(x,y,z,w) = (6s,2s,5s,3s)$,$(-2s,-6s,-3s,-5s)$ satisfy (10.1). In addition, we have many more solution patterns for (10.1). The process of obtaining the same is illustrated below:

Process 1

Taking

$$x = u+v, y = u-v, z = u+p, w = u-p \,, u \neq v, p \tag{10.2}$$

in (10.1) ,it results in second degree equation with equal terms and three variables

$$u^2 + 3v^2 = 28p^2 \tag{10.3}$$

Assume

$$p = a^2 + 3b^2 \tag{10.4}$$

Express the integer 28 in (10.3) as

$$28 = (5+i\sqrt{3})\,(5-i\sqrt{3}) \tag{10.5}$$

Using (10.4) & (10.5) in (10.3) & factorizing, consider

$$u + i\sqrt{3}v = (5+i\sqrt{3})\,(a+i\sqrt{3}b)^2 \tag{10.6}$$

Comparison of the coefficients of corresponding terms in (10.6) gives

$$\begin{aligned} u &= 5(a^2 - 3b^2) - 6ab, \\ v &= (a^2 - 3b^2) + 10ab. \end{aligned} \tag{10.7}$$

From (10.2) , one obtains

$$\begin{aligned} x &= 6\ (a^2 - 3b^2) + 4ab, \\ y &= 4\ (a^2 - 3b^2) - 16ab, \\ z &= 6\ (a^2 - 2b^2) - 6ab, \\ w &= 2\ (2a^2 - 9b^2) - 6ab\ . \end{aligned} \tag{10.8}$$

Note 1

In addition to (10.5), the integer 28 is written as

$$28 = (4 + i\,2\sqrt{3})\ (4 - i\,2\sqrt{3}).$$

In this case , a new set of solutions for (10.1) is obtained.

Process 2

Write (10.3) as

$$u^2 + 3v^2 = 28\,p^2 * 1 \tag{10.9}$$

Assume the integer 1 in (10.9) as

$$1 = \frac{(1 + i\sqrt{3})\ (1 - i\sqrt{3})}{4} \tag{10.10}$$

Using (10.4),(10.5) & (10.10) in (10.9) & applying the method of factorization, take

$$u + i\sqrt{3}\,v = \frac{(5 + i\sqrt{3})(a + i\sqrt{3}\,b)^2\,(1 + i\sqrt{3})}{2} \tag{10.11}$$

Comparison of respective terms in (10.11) gives

$$\begin{aligned} u &= (a^2 - 3b^2) - 18ab, \\ v &= 3\ (a^2 - 3b^2) + 2ab \end{aligned}$$

From (10.2) , (10.1) is satisfied by

$$x = 4\ (a^2 - 3b^2) - 16ab,$$
$$y = -2\ (a^2 - 3b^2) - 20ab,$$
$$z = 2a^2 - 18ab, \qquad (10.12)$$
$$w = -6b^2 - 18ab\ .$$

Note 2

Apart from (10.10), consider

$$\mathbf{1 = \frac{(3a^2 - b^2 + i\sqrt{3}\,2ab)\ (3a^2 - b^2 - i\sqrt{3}\,2ab)}{(3a^2 + b^2)^2}},$$

$$\mathbf{1 = \frac{(a^2 - 3b^2 + i\sqrt{3}\,2ab)\ (a^2 - 3b^2 - i\sqrt{3}\,2ab)}{(a^2 + 3b^2)^2}}$$

A similar analysis gives two more sets of integer solutions to (10.1) .

Process 3

The ratio form of (10.3) is

$$\frac{u + 5p}{p + v} = \frac{3(p - v)}{u - 5p} = \frac{\alpha}{\beta}\ , \beta \neq 0$$

Solving the above system of double equations ,we get

$$u = 5\alpha^2 + 6\alpha\beta - 15\beta^2\ ,$$
$$v = -\alpha^2 + 10\ \alpha\beta + 3\beta^2$$

and

$$p = \alpha^2 + 3\beta^2 \qquad (10.13)$$

In view of (10.2), observe that the solutions to (10.1) are found to be

$$x = 4\alpha^2 + 16\alpha\beta - 12\beta^2\ ,$$
$$y = 6\alpha^2 - 4\alpha\beta - 18\beta^2\ ,$$
$$z = 6\alpha^2 + 6\alpha\beta - 12\beta^2\ , \qquad (10.14)$$
$$w = 4\alpha^2 + 6\alpha\beta - 18\beta^2$$

along with (10.13) .

Note 3

Also ,represent (10.3) as follows :

$$\frac{u+5p}{p-v}=\frac{3(p+v)}{u-5p}=\frac{\alpha}{\beta}\ ,\beta\neq 0,$$

$$\frac{u+5p}{3(p-v)}=\frac{(p+v)}{u-5p}=\frac{\alpha}{\beta}\ ,\beta\neq 0\ ,$$

$$\frac{u+5p}{3(p+v)}=\frac{(p-v)}{u-5p}=\frac{\alpha}{\beta}\ ,\beta\neq 0\ .$$

A similar analysis leads to three more patterns of solutions in integers to (10.1).

Process 4

Assume (10.3) to be

$$3v^2 = 28p^2 - u^2 \tag{10.15}$$

Take

$$v = 28a^2 - b^2 \tag{10.16}$$

Consider 3 in (10.15) as

$$3 = (\sqrt{28}+5)\ (\sqrt{28}-5) \tag{10.17}$$

Substituting (10.16) & (10.17) in (10.15) and factorizing, one has

$$\sqrt{28}\,p + u = (\sqrt{28}+5)\ (\sqrt{28}\,a+b)^2$$

On comparison of respective terms ,one has

$$u = 5(28a^2 + b^2) + 56ab$$

and

$$p = 28a^2 + b^2 + 10ab \tag{10.18}$$

From (10.2) , the solutions to (10.1) are

$$\begin{aligned} x &= 168a^2 + 4b^2 + 56ab, \\ y &= 112a^2 + 6b^2 + 56ab, \\ z &= 6(28a^2 + b^2) + 66ab, \\ w &= 4(28a^2 + b^2) + 46ab \end{aligned} \qquad (10.19)$$

jointly with (10.18).

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Chapter 11

ON NON-HOMOGENEOUS QUINARY CUBIC EQUATION

11.1 Technical Procedure

The non-uniform cubic equation with five variables is

$$xy - zw = R^3 \tag{11.1}$$

The insertion

$$x = u + v, y = u - v, z = p + v, w = p - v, u \neq v \neq p \tag{11.2}$$

in (11.1) gives

$$u^2 - p^2 = R^3 \tag{11.3}$$

Solving (11.3) through different ways and utilizing (11.2), the integer solutions to (11.1) are obtained.

Way 1

Consider (11.3) as

$$\begin{aligned} u + p &= R^3, \\ u - p &= 1 \end{aligned} \tag{11.4}$$

After some algebra, it is seen that

$$u = \frac{R^3 + 1}{2}, p = \frac{R^3 - 1}{2}, \tag{11.5}$$

The choice

$$R = 2k + 1 \tag{11.6}$$

in (11.5), gives

$$u = 4k^3 + 6k^2 + 3k + 1,$$
$$p = 4k^3 + 6k^2 + 3k.$$

From (11.2), one obtains

$$\begin{aligned} x &= 4k^3 + 6k^2 + 3k + 1 + v\ , \\ y &= 4k^3 + 6k^2 + 3k + 1 - v\ , \\ z &= k^3 + 6k^2 + 3k + v\ , \\ w &= k^3 + 6k^2 + 3k - v \end{aligned} \tag{11.7}$$

Thus, (11.7) satisfies (11.1) jointly with (11.6) .

Note 1

Apart from (11.5) , there are values of u, p, R satisfying (11.3). Knowing these values, the integer solutions to (11.1) are obtained through employing (11.2). For simplicity and brevity, a few choices of solutions in integers to (11.1) are exhibited as follows:

Example 1

$$x = \frac{R(R+1)}{2} + v, y = \frac{R(R+1)}{2} - v,$$
$$z = \frac{R(R-1)}{2} + v, w = \frac{R(R-1)}{2} - v, R > 1$$

Example 2

$$x = n(n+2) + v, y = n(n+2) - v$$
$$z = n(n-2) + v, w = n(n-2) - v, R = 2n\ , n \neq 0, \pm 2$$

Example 3

$$x = n(2n+1) + v, y = n(2n+1) - v$$
$$z = n(2n-1) + v, w = n(2n-1) - v, R = 2n$$

Example 4

$$x = 2n^3 + 1 + v, y = 2n^3 + 1 - v,$$
$$z = 2n^3 - 1 + v, w = 2n^3 - 1 - v, R = 2n$$

Way 2

The option

$$p = kR \tag{11.8}$$

in (11.3) gives

$$u^2 = R^2(R + k^2) \tag{11.9}$$

Let

$$\alpha^2 = R + k^2 \tag{11.10}$$

By inspection,

$$R_0 = s^2 + 2sk \,, \alpha_0 = s + k \tag{11.11}$$

Consider the second solution to (11.10) as

$$R_1 = R_0 + h \,, \alpha_1 = h - \alpha_0 \tag{11.12}$$

where h is an unknown integer to be determined. Inserting (11.12) in (11.10) & simplifying gives

$$h = 2\alpha_0 + 1$$

From (11.12), we have

$$\alpha_1 = \alpha_0 + 1, R_1 = R_0 + 2\alpha_0 + 1$$

A similar procedure gives the n^th^ solution (α_t, R_t) for (11.10) as

$$\alpha_t = \alpha_0 + t = (s + k + t)$$
$$R_t = R_0 + 2t\alpha_0 + t^2 = s^2 + 2sk + 2t(s + k) + t^2 \tag{11.13}$$

From (11.8) and (11.9), we obtain

$$p_t = kR_t = k(s^2 + 2sk + 2t(s + k) + t^2)$$
$$u_t = (R_t * \alpha_t) = (s + k + t)(s^2 + 2sk + 2t(s + k) + t^2)$$

Using (11.2), (11.1) is satisfied by

$$x_t = [(s+k+t)(s^2+2sk+2t(s+k)+t^2]+v$$
$$y_t = [(s+k+t)(s^2+2sk+2t(s+k)+t^2]-v$$
$$z_t = k[s^2+2sk+2t(s+k)+t^2]+v$$
$$w_t = k[s^2+2sk+2t(s+k)+t^2]-v$$

jointly with R_t given by (11.13).

Way 3

The option

$$u = kR \quad (11.14)$$

in (11.3) gives

$$p^2 = R^2(k^2 - R) \quad (11.15)$$

Let

$$\alpha^2 = k^2 - R \quad (11.16)$$

By inspection, observe that (11.16) is satisfied by

$$R_0 = -(s^2+2s)k^2 \ , \alpha_0 = (s+1)k \quad (11.17)$$

Following the above analysis, the general solution (α_t, R_t) to (11.16) is given by

$$\alpha_t = \alpha_0 - t$$
$$R_t = R_0 + 2t\alpha_0 - t^2 \quad (11.18)$$

From (11.14) and (11.15), we obtain

$$u_t = kR_t = k(R_0 + 2t\alpha_0 - t^2)$$
$$p_t = (\alpha_t * R_t) = (\alpha_0 - t)(R_0 + 2t\alpha_0 - t^2)$$

Utilizing (11.2), (11.1) is satisfied by

$$x_t = [k(R_0 + 2t\alpha_0 - t^2)] + v$$
$$y_t = [k(R_0 + 2t\alpha_0 - t^2)] - v$$
$$z_t = [(\alpha_0 - t)(R_0 + 2t\alpha_0 - t^2)] + v$$
$$w_t = [(\alpha_0 - t)(R_0 + 2t\alpha_0 - t^2)] - v$$

jointly with R_t given by (11.18).

Way 4

The option

$$x = R * y, \; z = R * w \tag{11.19}$$

in (11.1) leads to the Pythagorean equation

$$y^2 = w^2 + R^2 \tag{11.20}$$

which is satisfied by

$$\begin{aligned} y &= a^2 + b^2, \\ w &= a^2 - b^2, \\ R &= 2ab, \qquad a > b > 0. \end{aligned} \tag{11.21}$$

From (11.19), we get

$$\begin{aligned} x &= (2ab)(a^2 + b^2), \\ z &= (2ab)(a^2 - b^2). \end{aligned} \tag{11.22}$$

Thus, (11.21) & (11.22) satisfy (11.1).

Way 5

The choice

$$x = R^2 * y, \; z = R^2 * w \tag{11.23}$$

in (11.1) leads to rectangular hyperbola

$$y^2 - w^2 = R \tag{11.24}$$

which is satisfied by

$$\begin{aligned} y &= k+1, \\ w &= k, \\ R &= 2k+1 \end{aligned} \qquad (11.25)$$

From (11.23), we have

$$\begin{aligned} x &= (2k+1)^2 (k+1), \\ z &= (2k+1)^2 k. \end{aligned} \qquad (11.26)$$

Thus, (11.25) & (11.26) satisfy (11.1).

Note:

It is worth mentioning that there are other choices of integer solutions to (11.24).

For example, on scrutiny, observe that (11.24) is satisfied by

$$\begin{aligned} y &= k+2, \\ w &= k, \\ R &= 4(k+1). \end{aligned}$$

And from (11.23), one obtains

$$\begin{aligned} x &= 16(k+1)^2 (k+2), \\ z &= 16(k+1)^2 k. \end{aligned}$$

The interested readers may search for other patterns of integer solution to (11.1) through employing the algebraic identities. For simplicity and brevity, after a few calculations, the patterns of solutions to (11.1) are observed:

Pattern 1:

$$\begin{aligned} x &= 16a^2A^2(A+a), \\ y &= A+a, \\ z &= 16a^2A^2(A-a), \\ w &= A-a, \\ R &= 4aA. \end{aligned}$$

Pattern 2:

$$x = (2aA + 1)^2 (A + a),$$
$$y = A + a,$$
$$z = (2aA + 1)^2 (A - a),$$
$$w = A,$$
$$R = 2aA + 1.$$

Pattern 3:

$$x = (a + b)^2 (2A + a - b)^2 (A + a),$$
$$y = (A + a),$$
$$z = (a + b)^2 (2A + a - b)^2 (A - b),$$
$$w = A - b,$$
$$R = (a + b)(2A + a - b).$$

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Chapter 12

A PORTRAYAL OF SOLUTIONS IN INTEGERS TO QUINARY THIRD DEGREE EQUATION

12.1 Technical Procedure

The Diophantine equation of degree three having five variables for solving is

$$x^3 + y^3 = 13(z + w)p^2 \tag{12.1}$$

Inserting

$$x = u + v, y = u - v, z = u + d, w = u - d, u \neq v \neq d \neq 0 \tag{12.2}$$

in (12.1),we get

$$u^2 + 3v^2 = 13p^2 \tag{12.3}$$

Solving (12.3) through different ways, we have varieties of solutions to (12.1).

Method I:

Assume

$$p = p(a,b) = a^2 + 3b^2 \tag{12.4}$$

Take integer 13 in (12.3) as

$$13 = (1 + i2\sqrt{3})(1 - i2\sqrt{3}) \tag{12.5}$$

Inserting (12.4) and (12.5) in (12.3) and factorizing, we write as the pair of equations as

$$u + i\sqrt{3}v = (1 + i2\sqrt{3})(a + i\sqrt{3}b)^2 = (1 + i2\sqrt{3})[f(a,b) + i\sqrt{3}g(a,b)]$$
$$u - i\sqrt{3}v = (1 - i2\sqrt{3})(a - i\sqrt{3}b)^2 = (1 - i2\sqrt{3})[f(a,b) - i\sqrt{3}g(a,b)]$$

where

$$f(a,b)=(a^2-3b^2)\ ,g(a,b)=2ab \tag{12.6}$$

On comparing the respective coefficients , one has

$$u=f(a,b)-6g(a,b)=a^2-3b^2-12ab$$
$$v=2f(a,b)+g(a,b)=2a^2-6b^2+2ab$$

From (12.2), (12.1) is satisfied by

$$x=x(a,b)=3a^2-9b^2-10ab\ ,$$
$$y=y(a,b)=-a^2+3b^2-14ab\ ,$$
$$z=z(a,b,d)=a^2-3b^2-12ab+d\ ,$$
$$w=w(a,b,d)=a^2-3b^2-12ab-d$$

jointly with (12.4) .

Note 1

The integer 13, apart from (12.5), may be factorized as

$$13=\frac{(7+i\sqrt{3})(7-i\sqrt{3})}{4}$$
$$13=\frac{(5+i3\sqrt{3})(5-i3\sqrt{3})}{4}$$

A similar process gives two more choices of solutions to (12.1).

Method 2

Assume (12.3) to be

$$u^2+3v^2=13p^2*1 \tag{12.7}$$

Consider

$$1=\frac{(a(s)+i\ b(s)\sqrt{3})(a(s)-i\,b(s)\sqrt{3})}{(a(s)+1)^2} \tag{12.8}$$

where

$$a(s)=(6s^2-6s+1),b(s)=(2s-1) \tag{12.9}$$

Substituting (12.4), (12.5) & (12.8) in (12.7) and employing factorization , we consider

$$(u+i\sqrt{3}v)=\left(1+i2\sqrt{3}\right)\left(a+i\sqrt{3}b\right)^2\frac{[a(s)+i\,b(s)\sqrt{3}]}{(a(s)+1)}$$

$$=\left(1+i2\sqrt{3}\right)[f(a,b)+i\sqrt{3}g(a,b)]\,\frac{[a(s)+i\,b(s)\sqrt{3}]}{(a(s)+1)}$$

$$=\{[f(a,b)-6g(a,b)]+i\sqrt{3}[2f(a,b)+g(a,b)]\}\frac{[a(s)+i\,b(s)\sqrt{3}]}{(a(s)+1)}$$

On comparing the respective terms, one has

$$\begin{aligned} u&=\frac{1}{(a(s)+1)}\{a(s)[f(a,b)-6g(a,b)]-3b(s)[2f(a,b)+g(a,b)]\} \\ v&=\frac{1}{(a(s)+1)}\{b(s)[f(a,b)-6g(a,b)]+a(s)[2f(a,b)+g(a,b)]\} \end{aligned} \tag{12.10}$$

Since, the focus is on finding integer solutions, taking

$$a=(a(s)+1)A,\, b=(a(s)+1)B$$

in (12.4) & (12.10) and utilizing (12.2) ,(12.1) is satisfied by:

$$\begin{aligned} x&=(a(s)+1)\{[a(s)+b(s)][f(A,B)-6g(A,B)]+[a(s)-3b(s)][2f(A,B)+g(A,B)]\}, \\ y&=(a(s)+1)\{[a(s)-b(s)][f(A,B)-6g(A,B)]-[a(s)+3b(s)][2f(A,B)+g(A,B)]\}, \\ z&=(a(s)+1)\{a(s)[f(A,B)-6g(A,B)]-3b(s)[2f(A,B)+g(A,B)]\}+d, \\ w&=(a(s)+1)\{a(s)[f(A,B)-6g(A,B)]-3b(s)[2f(A,B)+g(A,B)]\}-d, \\ p&=(a(s)+1)^2[A^2+3B^2]. \end{aligned}$$

(12.11)

To analyse the nature of solutions,one has to take particular values to s. For simplicity and clear understanding , the option s=1 gives

$$a(s)=a(1)=1,\, b(s)=b(1)=1$$

$$1=\frac{(1+i\sqrt{3})(1-i\sqrt{3})}{4}$$

Also ,

$$f(A,B)=A^2-3B^2,\ g(A,B)=2AB$$

From (12.11) ,the integer solutions to (12.1) are given by

$$x=x(A,B)=-4A^2+12B^2-56AB$$
$$y=y(A,B)=-16A^2+48B^2-16AB$$
$$z=z(A,B,d)=-10A^2+30B^2-36AB+d$$
$$w=w(A,B,d)=-10A^2+30B^2-36AB-d$$
$$p=p(A,B)=4A^2+12B^2$$

Note 2

Apart from (12.8) , one may have other Inspections to integer 1 which are exhibited below

Inspection 1:

$$1=\frac{(a(s)+i\,b(s)\sqrt{3})(a(s)-i\,b(s)\sqrt{3})}{(a(s)+2)^2}$$

where

$$a(s)=(3s^2-1),\ b(s)=2s \qquad (12.12)$$

Inspection 2:

$$1=\frac{(a(s)+i\,b(s)\sqrt{3})(a(s)-i\,b(s)\sqrt{3})}{(a(s)+6s^2)^2}$$

where

$$a(s)=(r^2-3s^2),\ b(s)=2rs \qquad (12.13)$$

Inspection 3:

$$1=\frac{(a(s)+i\,b(s)\sqrt{3})(a(s)-i\,b(s)\sqrt{3})}{(a(s)+2s^2)^2}$$

where

$$a(s)=(3r^2-s^2),\ b(s)=2rs \qquad (12.14)$$

Inspection 4:

$$1=\frac{(1+i\sqrt{3}\alpha_n)(1-i\sqrt{3}\alpha_n)}{(\beta_n)^2}=\frac{(2+i\ g_n)(2-i\ g_n)}{(f_n)^2}$$

where

$$\alpha_n=\frac{1}{2\sqrt{3}}g_n=\frac{1}{2\sqrt{3}}[(2+\sqrt{3})^{n+1}-(2-\sqrt{3})^{n+1}],$$
$$\beta_n=\frac{1}{2}f_n=\frac{1}{2}[(2+\sqrt{3})^{n+1}+(2-\sqrt{3})^{n+1}],n=0,1,2,... \quad (12.15)$$

Using (12.12), (12.13), (12.14) and (12.15) in (12.11) in turn ,the corresponding integer solutions to (12.1) are obtained.

Method 3

Consider (12.3) as

$$13p^2-3v^2=u^2*1 \quad (12.16)$$

Assume

$$u=13a^2-3b^2 \quad (12.17)$$

Write the integer 1 in (12.16) as

$$1=(\sqrt{13}+2\sqrt{3})(\sqrt{13}-2\sqrt{3}) \quad (12.18)$$

Substituting (12.17) & (12.18) in (12.16) and applying factorization , consider

$$\sqrt{13}p+\sqrt{3}v=(\sqrt{13}+2\sqrt{3})(\sqrt{13}a+\sqrt{3}b)^2$$

from which , on equating the corresponding terms ,one obtains

$$p=13a^2+3b^2+12ab,$$
$$v=2(13a^2+3b^2)+26ab \quad (12.19)$$

From (12.2), (12.1) is satisfied by

$$x = 39a^2 + 3b^2 + 26ab,$$
$$y = -13a^2 - 9b^2 - 26ab,$$
$$z = 13a^2 - 3b^2 + d,$$
$$w = 13a^2 - 3b^2 - d$$

jointly with p in (12.19).

Note 3

In addition to (12.18) , we have

$$1 = \frac{(2\sqrt{13} + 3\sqrt{3})\,(2\sqrt{13} - 3\sqrt{3})}{25}$$

The repetition of the above process leads to a different set of solutions to (12.1) .

Method 4

Rewrite (12.3) as

$$13p^2 - u^2 = 3v^2 \qquad (12.20)$$

Assume

$$v = 13a^2 - b^2 \qquad (12.21)$$

Write the integer 3 in (12.20) as

$$3 = (2\sqrt{13} + 7)(2\sqrt{13} - 7) \qquad (12.22)$$

Inserting (12.21) & (12.22) in (12.20) and using factorization, we consider

$$\sqrt{13}\,p + u = (2\sqrt{13} + 7)(\sqrt{13}\,a + b)^2$$

On comparing, one has

$$\begin{aligned} p &= 2(13a^2 + b^2) + 14ab, \\ u &= 7(13a^2 + b^2) + 52ab \end{aligned} \qquad (12.23)$$

From (12.2), (12.1) is satisfied by

$$x = 104a^2 + 6b^2 + 52ab,$$
$$y = 78a^2 + 8b^2 + 52ab,$$
$$z = 7(13a^2 + b^2) + 52ab + d,$$
$$w = 7(13a^2 + b^2) + 52ab - d$$

jointly with p in (12.23).

Note 4

Apart from (12.22) , we have

$$3 = \frac{(\sqrt{13}+1)(\sqrt{13}-1)}{4},$$
$$3 = \frac{(2\sqrt{13}+5)(2\sqrt{13}-5)}{9}$$

The repetition of the above process leads to different sets of solutions to (12.1) .

Method 5

Consider (12.20) as

$$13p^2 - u^2 = 3v^2 * 1 \qquad (12.24)$$

Write the integer 1 in (12.24) as

$$1 = (5\sqrt{13}+18)(5\sqrt{13}-18) \qquad (12.25)$$

Inserting (12.21) ,(12.22) & (12.25) in (12.24) and applying factorization , we consider

$$\sqrt{13}p + u = (2\sqrt{13}+7)(\sqrt{13}a + b)^2 (5\sqrt{13}+18)$$

On comparison, one has

$$p = 71(13a^2 + b^2) + 512ab,$$
$$u = 256(13a^2 + b^2) + 1846\,ab, \qquad (12.26)$$

From (12.2), (12.1) is satisfied by

$$x = 3341a^2 + 255b^2 + 1846ab,$$
$$y = 3315a^2 + 257b^2 + 1846ab,$$
$$z = 3328a^2 + 256b^2 + 1846ab + d,$$
$$w = 3328a^2 + 256b^2 + 1846ab - d$$

jointly with p in (12.26).

Note 5

In addition to (12.25), we have

$$1 = \frac{(\sqrt{13}+3)(\sqrt{13}-3)}{4}$$
$$1 = \frac{(\sqrt{13}+2)(\sqrt{13}-2)}{9}$$
$$1 = \frac{(5\sqrt{13}+1)(5\sqrt{13}-1)}{324}$$

The repetition of the above process leads to different sets of solutions to (12.1).

Method 6

Express (12.3) in the form of ratio as below:

$$\frac{u+p}{2p+v} = \frac{3(2p-v)}{u-p} = \frac{\alpha}{\beta}, \beta \neq 0 \qquad (12.27)$$

The above equation is written as the system of double equations

$$\beta u - \alpha v + (\beta - 2\alpha)p = 0$$
$$\alpha u + 3\beta v - (6\beta + \alpha)p = 0$$

Employing the method of cross-multiplication, we have

$$u = \alpha^2 - 3\beta^2 + 12\alpha\beta,$$
$$v = -2\alpha^2 + 6\beta^2 + 2\alpha\beta, \qquad (12.28)$$
$$p = \alpha^2 + 3\beta^2$$

From (12.2), (12.1) is satisfied by

$$x = -\alpha^2 + 3\beta^2 + 14\alpha\beta,$$
$$y = 3\alpha^2 - 9\beta^2 + 10\alpha\beta,$$
$$z = \alpha^2 - 3\beta^2 + 12\alpha\beta + d,$$
$$w = \alpha^2 - 3\beta^2 + 12\alpha\beta - d$$

jointly with p given in (12.28).

Note 6

Apart from (12.27), we may have the following ratio form representations

$$\frac{u+p}{3(2p+v)} = \frac{(2p-v)}{u-p} = \frac{\alpha}{\beta}, \beta \neq 0$$
$$\frac{u+p}{2p-v} = \frac{3(2p+v)}{u-p} = \frac{\alpha}{\beta}, \beta \neq 0$$
$$\frac{u+p}{3(2p-v)} = \frac{(2p+v)}{u-p} = \frac{\alpha}{\beta}, \beta \neq 0$$

Proceeding as above, three additional patterns to (12.1) are determined.

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Chapter 13

HOMOGENEOUS QUINARY CUBIC EQUATION

13.1 Technical Procedure

The uniform third quinary degree equation is

$$x^3 - y^3 = z^3 - w^3 + 90\, t^3 \tag{13.1}$$

Inserting

$$x = u + v, y = u - v, z = p + v, w = p - v, t = v, u \neq v \neq p \tag{13.2}$$

in (13.1) gives

$$u^2 = 15v^2 + p^2 \tag{13.3}$$

Solving (13.3) for getting the values of u, v, p and using (13.2), one obtains varieties of patterns to (13.1).

Way 1
Write (13.3) as

$$\frac{u+p}{15v} = \frac{v}{u-p} = \frac{A}{B}, B \neq 0 \tag{13.4}$$

Express (13.4) as two equations

$$Bu - 15Av + Bp = 0$$

$$Au - Bv - Ap = 0$$

Utilizing the process of cross multiplication, one has

$$u = 15A^2 + B^2, v = 2AB, p = 15A^2 - B^2$$

Thus, the integer solutions to (13.1) are given by

$$x = 15A^2 + B^2 + 2AB,$$
$$y = 15A^2 + B^2 - 2AB,$$
$$z = 15A^2 - B^2 + 2AB,$$
$$w = 15A^2 - B^2 - 2AB,$$
$$t = 2AB.$$

Remark 1

Apart from (13.4), we have

$$\frac{u+p}{5v} = \frac{3v}{u-p} = \frac{A}{B} \qquad (13.5)$$

satisfied by

$$u = 5A^2 + 3B^2, v = 2AB, p = 5A^2 - 3B^2$$

Thus,

$$x = 5A^2 + 3B^2 + 2AB,$$
$$y = 5A^2 + 3B^2 - 2AB,$$
$$z = 5A^2 - 3B^2 + 2AB,$$
$$w = 5A^2 - 3B^2 - 2AB,$$
$$t = 2AB.$$

satisfy (13.1).

Way 2

Write (13.3) as in Table -1:

Table-1 System of double equations

System	I	II	III	IV	V
$u+p$	$15v^2$	$5v^2$	$3v^2$	v^2	$15v$
$u-p$	1	3	5	15	v

The above pair of equations in Table 1 are solved for v, u, p and utilizing (13.2), the integer solutions to (13.1) is determined. For brevity and simplicity, the respective integer solutions are exhibited below:

Solutions from System I

$$x = 30k^2 + 32k + 9, y = 30k^2 + 28k + 7,$$
$$z = 30k^2 + 32k + 8, w = 30k^2 + 28k + 6, t = 2k + 1$$

Solutions from System II

$$x = 10k^2 + 12k + 5\ ,\ y = 10k^2 + 8k + 3,$$
$$z = 10k^2 + 12k + 2, w = 10k^2 + 8k, t = 2k + 1$$

Solutions from System III

$$x = 6k^2 + 8k + 5\ ,\ y = 6k^2 + 4k + 3,$$
$$z = 6k^2 + 8k, w = 6k^2 + 4k - 2, t = 2k + 1$$

Solutions from System IV

$$x = 2k^2 + 4k + 9\ ,\ y = 2k^2 + 7,$$
$$z = 2k^2 + 4k - 6, w = 2k^2 - 8\ , t = 2k + 1$$

Solutions from System V

$$x = 9k, y = 7k, z = 8k, w = 6k, t = k$$

Way 3

Write (13.3) as

$$p^2 + 15v^2 = u^2 * 1 \tag{13.6}$$

Take

$$u = a^2 + 15b^2 \tag{13.7}$$

Express the integer 1 in (13.6) as

$$1 = \frac{(1 + i\sqrt{15})\ (1 - i\sqrt{15})}{16} \tag{13.8}$$

Substituting (13.7) & (13.8) in (13.6) and applying factorization, consider

$$\begin{aligned} p + i\sqrt{15}\,v &= \frac{(1 + i\sqrt{15})}{4}\ (a + i\sqrt{15}\,b)^2 \\ &= \frac{(1 + i\sqrt{15})}{4}\ [f(a,b) + i\sqrt{15}\,g(a,b)] \end{aligned} \tag{13.9}$$

where

$$f(a,b) = a^2 - 15b^2\ , g(a,b) = 2ab \tag{13.10}$$

On comparing

$$\begin{aligned} p &= \frac{f(a,b) - 15g(a,b)}{4} \\ v &= \frac{f(a,b) + g(a,b)}{4} \end{aligned} \tag{13.11}$$

As the aim is to obtain integer solutions, replacing a by $2A$ & b by $2B$ in (13.7) and (13.11), we have

$$u = 4A^2 + 60B^2\ ,$$
$$p = f(A,B) - 15g(A,B)\ ,$$
$$v = f(A,B) + g(A,B)\ .$$

From (13.2), (13.1) is satisfied by

$$x = 5A^2 + 45B^2 + 2AB,$$
$$y = 3A^2 + 75B^2 - 2AB,$$
$$z = 2A^2 - 30B^2 - 28AB,$$
$$w = -32AB,$$
$$t = A^2 - 15B^2 + 2AB.$$

Note 1

Also, 1 in (13.6) has the following representations:

(i) $$1 = \frac{(15r^2 - s^2 + i\,2rs\sqrt{15})\,(15r^2 - s^2 - i\,2rs\sqrt{15})}{(15r^2 + s^2)^2}$$

(ii) $$1 = \frac{(30s^2 - 30s + 7 + i(2s-1)\sqrt{15})\,(30s^2 - 30s + 7 - i(2s-1)\sqrt{15})}{(30s^2 - 30s + 8)^2}$$

(iii) $$1 = \frac{(6s^2 - 6s - 1 + i(2s-1)\sqrt{15})\,(6s^2 - 6s - 1 - i(2s-1)\sqrt{15})}{(6s^2 - 6s + 4)^2}$$

(iv) $$1 = \frac{(2s^2 - 2s - 7 + i(2s-1)\sqrt{15})\,(2s^2 - 2s - 7 - i(2s-1)\sqrt{15})}{(2s^2 - 2s + 8)^2}$$

$$1 = \frac{(2 + i g_n)\,(2 - i g_n)}{(f_n)^2},$$

(v) $$f_n = (4 + \sqrt{15})^{n+1} + (4 - \sqrt{15})^{n+1},$$
$$g_n = (4 + \sqrt{15})^{n+1} - (4 - \sqrt{15})^{n+1}, n = 0,1,2,...$$

A similar analysis gives five additional choices of solutions to (13.1).

Way 4

Write (13.3) as

$$u^2 - 15v^2 = p^2 * 1 \tag{13.12}$$

Assume

$$p = a^2 - 15b^2 \tag{13.13}$$

Consider the integer 1 in (13.12) as

$$1 = (4 + \sqrt{15})\,(4 - \sqrt{15}) \tag{13.14}$$

Substituting (13.13) & (13.14) in (13.12) and utilizing factorization, consider

$$\begin{aligned} u + \sqrt{15}\,v &= (4 + \sqrt{15})\,(a + \sqrt{15}\,b)^2 \\ &= (4 + \sqrt{15})\,[(a^2 + 15b^2) + 2ab\sqrt{15}] \end{aligned} \tag{13.15}$$

Equating the rational and irrational parts in (13.15), one obtains

$$u = 4(a^2 + 15b^2) + 30ab\ ,$$
$$v = (a^2 + 15b^2) + 8ab$$

In view of (13.2),

$$\begin{aligned} x &= 5(a^2 + 15b^2) + 38ab\ , \\ y &= 3(a^2 + 15b^2) + 22ab\ , \\ z &= 2a^2 + 8ab, \\ w &= -30b^2 - 8ab, \\ t &= a^2 + 15b^2 + 8ab\ . \end{aligned} \tag{13.16}$$

satisfy (13.1).

Note 2

Also, the integer 1 in (13.12) has the following representations:

(i) $$1 = \frac{(30s^2 - 30s + 8 + (2s-1)\sqrt{15})\ (30s^2 - 30s + 8 - (2s-1)\sqrt{15})}{(30s^2 - 30s + 7)^2}$$

(ii) $$1 = \frac{(6s^2 - 6s + 4 + (2s-1)\sqrt{15})\ (6s^2 - 6s + 8 - (2s-1)\sqrt{15})}{(6s^2 - 6s - 1)^2}$$

(iii) $$1 = \frac{(15r^2 + s^2 + 2rs\sqrt{15})\ (15r^2 + s^2 - 2rs\sqrt{15})}{(15r^2 - s^2)^2}$$

A similar analysis gives three additional choices of solutions to (13.1).

Way 5

Consider (3) as

$$15v^2 + p^2 = u^2 * 1 \tag{13.17}$$

Take

$$u = 15a^2 + b^2 \tag{13.18}$$

Express the integer 1 in (13.17) as

$$1 = \frac{(\sqrt{15} + i)\ (\sqrt{15} - i)}{16} \tag{13.19}$$

Substituting (13.18) & (13.19) in (13.17) and applying factorization , consider

$$\begin{aligned} \sqrt{15}\,v + i\,p &= \frac{(\sqrt{15} + i)}{4}\ (\sqrt{15}\,a + i\,b)^2 \\ &= \frac{(\sqrt{15} + i)}{4}\ [f(a,b) + i\sqrt{15}\,g(a,b)] \end{aligned} \tag{13.20}$$

where

$$f(a,b)=15a^2-b^2, g(a,b)=2ab \tag{13.21}$$

On comparing, we have

$$p=\frac{f(a,b)+15g(a,b)}{4}$$
$$v=\frac{f(a,b)-g(a,b)}{4} \tag{13.22}$$

Replacing a by $2A$ & b by $2B$ in (13.18) and (13.22) , we have

$$u=60A^2+4B^2,$$
$$p=f(A,B)+15g(A,B),$$
$$v=f(A,B)-g(A,B).$$

From (13.2), 13.1) is satisfied by

$$x=75A^2+3B^2-2AB,$$
$$y=45A^2+5B^2+2AB,$$
$$z=30A^2-2B^2+28AB,$$
$$w=32AB,$$
$$t=15A^2-B^2-2AB.$$

Note 3

In addition to (19) , the integer 1 in (13.17) has the following representations :

(i) $$1=\frac{(2rs\sqrt{15}+i(r^2-15s^2))\,(2rs\sqrt{15}-i(r^2-15s^2))}{(r^2+15s^2)^2}$$

(ii) $$1=\frac{((2s-1)\sqrt{15}+i\,(30s^2-30s+7))\,((2s-1)\sqrt{15}-i\,(30s^2-30s+7))}{(30s^2-30s+8)^2}$$

A similar process leads to two additional patterns to (13.1) .

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Chapter 14

NON-HOMOGENEOUS CUBIC WITH SIX UNKNOWNS

14.1 Technical Procedure

The third degree non-uniform polynomial equation having six unknowns is

$$xy - zw + 2T^2 = (a^2 + b^2)R^3 \tag{14.1}$$

Inserting

$$x = u + v, y = u - v, z = P + v, w = P - v, T = P, u \neq v \neq P \neq 0 \tag{14.2}$$

in (14.1) gives

$$u^2 + P^2 = (a^2 + b^2)\, R^3 \tag{14.3}$$

The process of obtaining patterns of integer solutions (14.1) is illustrated below:

Process 1

It is seen that

$$\begin{aligned} u &= (a^2 + b^2)^2\, m(m^2 + n^2)\,, \\ P &= (a^2 + b^2)^2\, n(m^2 + n^2)\,, \\ R &= (a^2 + b^2)\,(m^2 + n^2). \end{aligned} \tag{14.4}$$

satisfy (14.3).

Thus, one obtains integer solutions to (14.1) as

$$x=(a^2+b^2)^2\, m(m^2+n^2)+v,$$
$$y=(a^2+b^2)^2\, m(m^2+n^2)-v,$$
$$z=(a^2+b^2)^2\, n(m^2+n^2)+v,$$
$$w=(a^2+b^2)^2\, n(m^2+n^2)-v,$$
$$T=(a^2+b^2)^2\, n(m^2+n^2)$$

jointly with the value of R in (14.4) . Note that v is an arbitrary non-zero integer for obtaining distinct solutions. It is worth to mention that, if v=0, then, the solutions for the equation $x^2-z^2+2T^2=(a^2+b^2)\,R^3$ are obtained.

Process 2

Assume

$$R=A^2+B^2 \qquad (14.5)$$

Substituting (14.5) in (14.3) and employing factorization ,consider

$$\begin{aligned} u+iP &= (a+ib)\,(A+iB)^3 \\ &= (a+ib)\,[f(A,B)+i\,g(A,B)] \end{aligned} \qquad (14.6)$$

where

$$\begin{aligned} f(A,B) &= A^3-3AB^2, \\ g(A,B) &= 3A^2B-B^3. \end{aligned} \qquad (14.7)$$

On comparison in (14.6) , observe

$$\begin{aligned} u &= a\,f(A,B)-b\,g(A,B), \\ P &= b\,f(A,B)+a\,g(A,B). \end{aligned} \qquad (14.8)$$

From (14.2)

$$\begin{aligned} x &= a\,f(A,B)-b\,g(A,B)+v, \\ y &= a\,f(A,B)-b\,g(A,B)-v, \\ z &= b\,f(A,B)+a\,g(A,B)+v, \\ w &= b\,f(A,B)+a\,g(A,B)-v, \\ T &= b\,f(A,B)+a\,g(A,B). \end{aligned} \qquad (14.9)$$

Thus , (14.5) & (14.9) satisfy (14.1) .

Process 3

Let

$$F(a,b,p,q) = a(p^2 - q^2) + b(2pq),$$
$$G(a,b,p,q) = a(2pq) - b(p^2 - q^2) \qquad (14.10)$$

where p,q are non-zero distinct integers.

Observe that

$$[F(a,b,p,q) + iG(a,b,p,q)]\,[F(a,b,p,q) - iG(a,b,p,q)] = F^2(a,b,p,q) + G^2(a,b,p,q)$$
$$= (a^2 + b^2)\,(p^2 + q^2)^2$$

Thus , we get

$$(a^2 + b^2) = \frac{[F(a,b,p,q) + iG(a,b,p,q)]\,[F(a,b,p,q) - iG(a,b,p,q)]}{(p^2 + q^2)^2} \qquad (14.11)$$

Substituting (14.5) & (14.11) in (14.3) and following the procedure as in Process 2 , one has

$$u = \frac{[f(A,B)F(a,b,p,q) - g(A,B)G(a,b,p,q)]}{(p^2 + q^2)},$$
$$P = \frac{[g(A,B)F(a,b,p,q) + f(A,B)G(a,b,p,q)]}{(p^2 + q^2)} \qquad (14.12)$$

Replacing A by $(p^2 + q^2)\,U$ and B by $(p^2 + q^2)\,V$ in (14.5) & (14.12) ,we have

$$R = (p^2 + q^2)^2 (U^2 + V^2)$$
$$u = (p^2 + q^2)^2 [f(U,V)F(a,b,p,q) - g(U,V)G(a,b,p,q)], \qquad (14.13)$$
$$P = (p^2 + q^2)^2 [g(U,V)F(a,b,p,q) + f(U,V)G(a,b,p,q)]$$

From (14.2), (14.1) is satisfied by

$$x = (p^2 + q^2)^2 [f(U,V)F(a,b,p,q) - g(U,V)G(a,b,p,q)] + v,$$
$$y = (p^2 + q^2)^2 [f(U,V)F(a,b,p,q) - g(U,V)G(a,b,p,q)] - v,$$
$$z = (p^2 + q^2)^2 [g(U,V)F(a,b,p,q) + f(U,V)G(a,b,p,q)] + v, \qquad (14.14)$$
$$w = (p^2 + q^2)^2 [g(U,V)F(a,b,p,q) + f(U,V)G(a,b,p,q)] - v,$$
$$T = (p^2 + q^2)^2 [g(U,V)F(a,b,p,q) + f(U,V)G(a,b,p,q)]$$

jointly with R given in (14.13).

Illustration:

Take

$$p = 2 \ ,q = 1, \ a = 2, \ b = 1, U = 3, \ V = 2, \ v = 1$$
$$f(U,V) = f(3,2) = -9$$
$$g(U,V) = g(3,2) = 46$$
$$F(a,b,p,q) = F(2,1,2,1) = 10$$
$$G(a,b,p,q) = G(2,1,2,1) = 5$$

From (14.13) and (14.14), we get,

$$R = 325, \ x = -7999, \ y = -8001, z = 10376, \ w = 10374, T = 10375$$

which satisfy

$$x\,y - z\,w + 2T^2 = 5R^3$$

Process 4

The insertion of the transformations

$$\begin{aligned} P &= (a^2 + b^2)^2 \, k\,S, \\ R &= (a^2 + b^2)\, S \end{aligned} \qquad (14.15)$$

in (14.3) leads to

$$u^2 = (a^2 + b^2)^4 \, S^2 \, (S - k^2) \qquad (14.16)$$

After performing some algebra, it is seen that the R.H.S. of (14.16) is a perfect square for values of S given by

$$S_t = (c\,k + t)^2 + k^2 \ , t = 0,1,2,...$$

and thus we have

$$u = u_t = (a^2 + b^2)^2 \, (c\,k + t)[(c\,k + t)^2 + k^2]$$

From (14.15) ,one obtains

$$\begin{aligned} P = P_t &= (a^2 + b^2)^2 \, k\,S_t = k\,(a^2 + b^2)^2 \, [(c\,k + t)^2 + k^2], \\ R = R_t &= (a^2 + b^2)\, S_t = (a^2 + b^2)\, [(c\,k + t)^2 + k^2] \end{aligned} \qquad (14.17)$$

In view of (14.2) ,the integer solutions to (14.1) are given by

$$x = x_t = u_t + v = (a^2+b^2)^2 (ck+t)[(ck+t)^2 + k^2] + v,$$
$$y = y_t = u_t - v = (a^2+b^2)^2 (ck+t)[(ck+t)^2 + k^2] - v,$$
$$z = z_t = P_t + v = k(a^2+b^2)^2 [(ck+t)^2 + k^2] + v ,$$
$$w = w_t = P_t - v = k(a^2+b^2)^2 [(ck+t)^2 + k^2] - v ,$$
$$T = T_t = P_t = k(a^2+b^2)^2 [(ck+t)^2 + k^2]$$

jointly with $R = R_t$ given in (14.17) .

Process 5

Inserting

$$x = u+v, y = u-v, z = P+v, w = P-v, T = 5P, u \neq v \neq P \neq 0 \tag{14.18}$$
in (14.1) gives

$$u^2 + 49P^2 = (a^2+b^2)\ R^3 \tag{14.19}$$

By inspection, observe that (14.19) is satisfied by

$$\begin{aligned} u &= (a^2+b^2)^2\, m(m^2+49n^2)\ , \\ P &= (a^2+b^2)^2\, n(m^2+49n^2)\ , \\ R &= (a^2+b^2)\ (m^2+49n^2). \end{aligned} \tag{14.20}$$

From (14.18),

$$x = (a^2+b^2)^2\, m(m^2+49n^2) + v,$$
$$y = (a^2+b^2)^2\, m(m^2+49n^2) - v,$$
$$z = (a^2+b^2)^2\, n(m^2+49n^2) + v,$$
$$w = (a^2+b^2)^2\, n(m^2+49n^2) - v,$$
$$T = (a^2+b^2)^2\, n(m^2+49n^2)$$

satisfy (14.1) jointly with (14.10).

Process 6:

Assume

$$R = 49\,[A^2 + B^2] \tag{14.21}$$

Substituting (14.21) in (14.19) & employing factorization, consider

$$u + iP = 7^3 (a + ib)\,(A + iB)^3$$
$$= 7^3 (a + ib)\,[f(A,B) + ig(A,B)]$$

Following the procedure as in process 2, we have,

$$\begin{aligned} u &= 7^3 [a f(A,B) - b g(A,B)], \\ P &= 7^2 [b f(A,B) + a g(A,B)]. \end{aligned} \qquad (14.22)$$

From (14.22) and (14.18) , one has the integer solutions to (14.1) as

$$\begin{aligned} x &= 7^3 [a f(A,B) - b g(A,B)] + v, \\ y &= 7^3 [a f(A,B) - b g(A,B)] - v, \\ z &= 7^2 [b f(A,B) + a g(A,B)] + v, \\ w &= 7^2 [b f(A,B) + a g(A,B)] - v, \\ T &= 5 * 7^2 [b f(A,B) + a g(A,B)]. \end{aligned} \qquad (14.23)$$

jointly with R in (14.21).

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Chapter 15

ON UNIFORM THIRD DEGREE EQUATION HAVING SIX PARAMETERS

15.1 Technical Procedure

The uniform third degree equation with six variables is

$$(w^2+p^2-z^2)(w-p)=(k^2+2)(x+y)R^2 \quad (15.1)$$

Inserting

$$x=v+1, y=v-1, z=u, w=u+v, p=u-v, u\neq v\neq \pm 1 \quad (15.2)$$

in (15.1) leads to

$$u^2+2v^2=(k^2+2)R^2 \quad (15.3)$$

Solving (15.3) is, we have patterns of solutions to (15.1).

Way 1:

By scrutiny,

$$u=k(k^2+2), v=(k^2+2), R=(k^2+2) \quad (15.4)$$

satisfies (15.3),

Thus, (15.1) is satisfied by

$$x=k^2+3, y=k^2+1, z=k(k^2+2),$$
$$w=(k+1)(k^2+2), p=(k-1)(k^2+2)$$

jointly with R given in (15.4) .

Way 2:

Write (15.3) as

$$(k^2+2)R^2-2v^2=u^2=u^2*1 \tag{15.5}$$

Assume

$$u=(k^2+2)a^2-2b^2 \tag{15.6}$$

Take integer 1 in (15.5) to be

$$1=\frac{(\sqrt{k^2+2}+\sqrt{2})(\sqrt{k^2+2}-\sqrt{2})}{k^2} \tag{15.7}$$

Inserting (15.6) & (15.7) in (15.5) & utilizing factorization , consider

$$\sqrt{k^2+2}\,R+\sqrt{2}\,v=\frac{(\sqrt{k^2+2}+\sqrt{2})(\sqrt{k^2+2}\,a+\sqrt{2}\,b)^2}{k}$$

On comparing, the coefficients of corresponding terms ,note that

$$R=\frac{(k^2+2)a^2+2b^2+4ab}{k},v=\frac{(k^2+2)a^2+2b^2+2(k^2+2)ab}{k} \tag{15.8}$$

Taking $a=kA, b=kB$ in (15.6) & (15.8) and using (15.2), (15.1) is satisfied by

$$x=k[(k^2+2)A^2+2B^2+2(k^2+2)AB]+1,$$
$$y=k[(k^2+2)A^2+2B^2+2(k^2+2)AB]-1,$$
$$z=k^2[(k^2+2)A^2-2B^2],$$
$$w=(k^2+k)(k^2+2)A^2+2B^2(k-k^2)+2k(k^2+2)AB,$$
$$p=(k^2-k)(k^2+2)A^2-2B^2(k+k^2)-2k(k^2+2)AB,$$
$$R=k[(k^2+2)A^2+2B^2+4AB]$$

Way 3:

Write (15.3) as

$$(k^2+2)R^2-u^2=2v^2 \tag{15.9}$$

Assume v as

$$v=(k^2+2)a^2-b^2 \qquad (15.10)$$

Take the integer 2 in (15.9) as

$$2=(\sqrt{k^2+2}+k)(\sqrt{k^2+2}-k) \qquad (15.11)$$

Following the procedure as in Way 2,the corresponding integer solutions to (15.1) are given by

$$x=(k^2+2)a^2-b^2+1, y=(k^2+2)a^2-b^2-1, z=k(k^2+2)a^2+k\,b^2+2(k^2+2)ab,$$
$$w=(k+1)(k^2+2)a^2+(k-1)\,b^2+2(k^2+2)ab, p=(k-1)(k^2+2)a^2+(k+1)\,b^2+2(k^2+2)ab,$$
$$R=(k^2+2)a^2+b^2+2kab$$

Way 4:

Rewrite (15.3) as

$$u^2=(k^2+2)\,R^2-2v^2 \qquad (15.12)$$

Introducing the linear transformations

$$R=X+2T, v=X+(k^2+2)T, u=kU \qquad (15.13)$$

in (15.12), one has

$$X^2=2(k^2+2)T^2+U^2 \qquad (15.14)$$

which is equivalent to the pair of equations as in case(a) & case(b) .

Case (a):

$$X+U=2(k^2+2)\,T$$
$$X-U=T$$

On solving, one has

$$X=\frac{(2k^2+5)\,T}{2},$$
$$U=\frac{(2k^2+3)\,T}{2}$$

Taking $T=2s$, we get

$$X = (2k^2 + 5)s,$$
$$U = (2k^2 + 3)s$$

Inserting (15.13) and using (15.2) ,we get the non-trivial integer solutions of (15.1) to be

$$x = (4k^2 + 9)s + 1,$$
$$y = (4k^2 + 9)s - 1,$$
$$z = (2k^3 + 3k)\ s,$$
$$R = (2k^2 + 9)s\ ,$$
$$w = (2k^3 + 4k^2 + 3k + 9)s,$$
$$p = (2k^3 - 4k^2 + 3k - 9)s.$$

Case (b):

$$X + U = (k^2 + 2)\ T$$
$$X - U = 2T$$

On solving, one has

$$X = \frac{(k^2 + 4)\ T}{2}\ ,$$
$$U = \frac{k^2\, T}{2}$$

Taking $T = 2s$, we get

$$X = (k^2 + 4)s,$$
$$U = k^2\, s$$

From (15.13) & (15.2) , (15.1) is satisfied by

$x = (3k^2 + 8)s + 1,$

$y = (3k^2 + 8)s - 1,$

$z = k^3 \, s,$

$R = (k^2 + 8)s \, ,$

$w = (k^3 + 3k^2 + 8)s,$

$p = (k^3 - 3k^2 - 8)s.$

Way 5:

Let

$$R = a^2 + 2b^2 \tag{15.15}$$

Take $(k^2 + 2)$ as

$$(k^2 + 2) = (k + i\sqrt{2})(k - i\sqrt{2}) \tag{15.16}$$

Applying (15.15) & (15.16) to (15.3), consider

$$\begin{aligned} u + i\sqrt{2}v &= (k + i\sqrt{2})\,(a + i\sqrt{2}b)^2 \\ &= (k + i\sqrt{2})\,[f(a,b) + i\,\sqrt{2}\,g(a,b)] \end{aligned} \tag{15.17}$$

where

$$f(a,b) = (a^2 - 2b^2)\,, g(a,b) = 2ab$$

On comparing (15.17),we get

$$u = k\,f(a,b) - 2\,g(a,b) = k(a^2 - 2b^2) - 4ab$$
$$v = f(a,b) + k\,g(a,b) = a^2 - 2b^2 + 2kab$$

From (15.2), (15.1) is satisfied by

$$x = a^2 - 2b^2 + 2kab + 1$$
$$y = a^2 - 2b^2 + 2kab - 1$$
$$z = ka^2 - 2kb^2 - 4ab$$
$$w = (a^2 - 2b^2)(k+1) + 2ab(k-2)$$
$$p = (a^2 - 2b^2)(k-1) - 2ab(k+2)$$

jointly with R given in (15.15) .

Way 6

Consider (15.3) as

$$u^2 + 2v^2 = (k^2 + 2)\ R^2 * 1 \qquad (15.18)$$

Write the integer 1 in (15.18) as

$$1 = \frac{(1 + i\,2\sqrt{2})(1 - i\,2\sqrt{2})}{9} \qquad (15.19)$$

Assume

$$R = 9(a^2 + 2b^2) \qquad (15.20)$$

Substituting (15.16) ,(15.19) & (15.20) in (15.18) and applying factorization , consider

$$u + i\sqrt{2}\,v = (k + i\sqrt{2})\ 9(a + i\sqrt{2}\,b)^2\ \frac{(1 + i\,2\sqrt{2})}{3}$$
$$= 3[k - 4 + i\sqrt{2}(1 + 2k)]\ [f(a,b) + i\sqrt{2}\,g(a,b)]$$

On comparing coefficients of corresponding terms , we have

$$u = 3[(k-4)f(a,b) - 2(1+2k)g(a,b)],$$
$$v = 3[(1+2k)f(a,b) + (k-4)g(a,b)].$$

From (15.2) ,(15.1) is satisfied by

$$x = 3[(1+2k)f(a,b) + (k-4)g(a,b)] + 1,$$
$$y = 3[(1+2k)f(a,b) + (k-4)g(a,b)] - 1,$$
$$z = 3[(k-4)f(a,b) - 2(1+2k)g(a,b)], \quad j$$
$$w = 3[(3k-3)f(a,b) - (3k+6)\ g(a,b)],$$
$$p = 3[(-k-5)f(a,b) + (2-5k)g(a,b)]$$

jointly with R given by (15.20) .

Note 1

Apart from (15.19) , the integer 1 may be expressed as below:

$$\mathbf{1 = \frac{(2r^2 - s^2 + i\sqrt{2}(2rs))\ (2r^2 - s^2 - i\sqrt{2}(2rs))}{(2r^2 + s^2)^2}}\ ,$$

$$\mathbf{1 = \frac{(r^2 - 2s^2 + i\sqrt{2}(2rs))\ (r^2 - 2s^2 - i\sqrt{2}(2rs))}{(r^2 + 2s^2)^2}}\ ,$$

$$\mathbf{1 = \frac{(2 + i h_n)\ (2 - i h_n)}{(e_n)^2}\ , n = 0,1,2,...}$$

where

$$\mathbf{e_n = (3 + 2\sqrt{2})^{n+1} + (3 - 2\sqrt{2})^{n+1}}\ ,$$
$$\mathbf{h_n = (3 + 2\sqrt{2})^{n+1} - (3 - 2\sqrt{2})^{n+1}}\ .$$

Following the above procedure, three more sets of integer solutions to (15.1) are obtained.

Way 7

Consider (15.3) as

$$\frac{(u + kR)}{(R + v)} = \frac{2\ (R - v)}{(u - kR)} = \frac{\alpha}{\beta}, \beta \neq 0 \qquad (15.21)$$

which is expressed as

$$u\ \beta - v\ \alpha + R(k\beta - \alpha) = 0$$
$$-u\ \alpha - 2\ v\ \beta + R\ (2\ \beta + k\alpha) = 0$$

whose solutions are

$$\begin{aligned} u &= -k\alpha^2 + 2k\beta^2 - 4\alpha\beta \\ v &= \alpha^2 - 2\beta^2 - 2k\alpha\beta \\ R &= -\alpha^2 - 2\beta^2 \end{aligned} \tag{15.22}$$

Thus, (15.1) is satisfied by

$$\begin{aligned} x &= \alpha^2 - 2\beta^2 - 2k\alpha\beta + 1 \\ y &= \alpha^2 - 2\beta^2 - 2k\alpha\beta - 1 \\ z &= -k\alpha^2 + 2k\beta^2 - 4\alpha\beta \\ w &= (1-k)\,\alpha^2 + 2\,(k-1)\,\beta^2 - (2k+4)\,\alpha\,\beta \\ p &= -(k+1)\,\alpha^2 + (2k+2)\,\beta^2 + (2k-4)\,\alpha\,\beta \end{aligned}$$

jointly with the value of R in (15.22) .

Note 2

Also, consider (15.21) as below

$$\text{(i)}\quad \frac{(u+kR)}{2(R+v)} = \frac{(R-v)}{(u-kR)} = \frac{\alpha}{\beta}$$

$$\text{(ii)}\quad \frac{(u+kR)}{(R-v)} = \frac{2(R-v)}{(u-kR)} = \frac{\alpha}{\beta}$$

A similar procedure as in Way 7, one obtains the respective solutions to (15.1) .

Conclusion:

This book is intended for advanced undergraduate and graduate scholars as well as researchers. The aim of this book is to give the readers an opportunity to learn and to know interesting results in the theory of Diophantine equations. We hope that scholars, researchers & anyone with an interest in number patterns and equations may be motivated to search for new techniques in solving multidegree and multivariate polynomial as well as transcendental Diophantine equations for their solutions in real integers. No doubt that the subject of Diophantine equations (polynomial and transcendental) is a treasure house and the analysis for getting integer solutions is a treasure hunt.

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