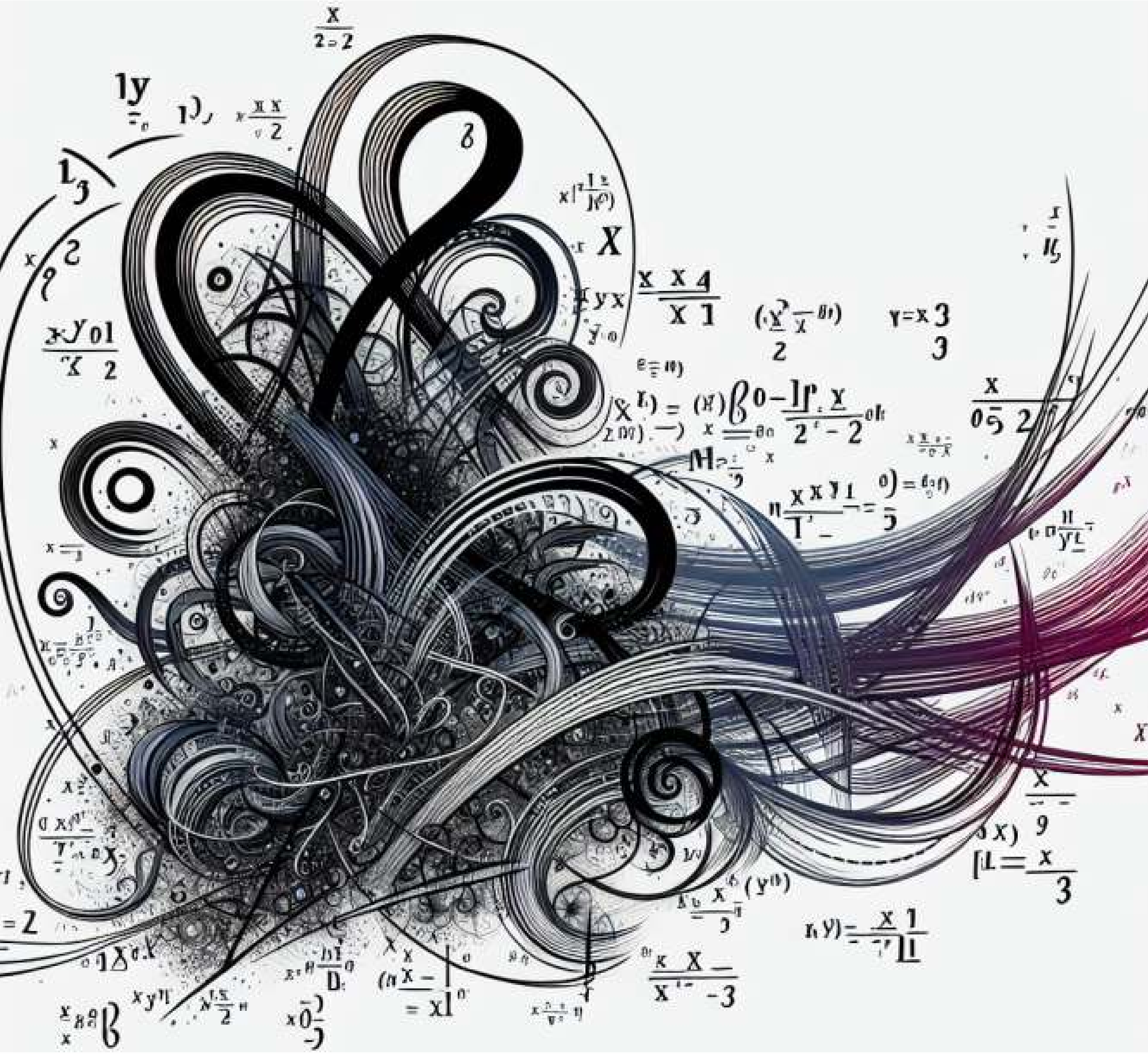

A Troupe of Special Second Degree Multivariable Polynomial Diophantine Equations with Integer Solutions

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Preface

The fascinating branch of Mathematics is the Theory of Numbers in which the subject of Diophantine equations requiring only the integer solutions is an interesting area to various mathematicians and to the lovers of mathematics because it is a treasure house in which the search for many hidden connections is a treasure hunt. In other words, the theory of Diophantine equations is an ancient subject that typically involves solving, polynomial equation in two or more variables or a system of polynomial equations with the number of unknowns greater than the number of equations, in integers and occupies a pivotal role in the region of mathematics. It is worth to mention that the Diophantine problems are plenty playing a significant role in the development of mathematics because the beauty of Diophantine equations is that the number of equations is less than the number of unknowns.

The theory of Diophantine equations provides a fertile ground for both professionals and amateurs. In addition to known results, the theory of Diophantine equations abounds with unsolved problems (Carmichael.,1959; Dickson.,1952; Mordell.,1969). In this context, for simplicity and brevity, one may refer (Gopalan et.al., 2012, 2015, 2021, 2024; Mahalakshmi, Shanthi., 2023; Sathiyapriya et.al., 2024; Shanthi.,2023; Shanthi, Mahalakshmi.,2023; Shanthi, Gopalan.,2024; Thiruniraiselvi, Gopalan., 2024; Vidhyalakshmi et.al., 2022) for some binary and ternary quadratic Diophantine equations. Although many of its results can be stated in simple and elegant terms, their proofs are sometimes long and complicated. Many unsolved problems that have been daunting mathematicians for centuries provide unlimited opportunities to expand the frontiers of mathematical knowledge. The subject of Diophantine equations has fascinated and inspired both amateurs and mathematicians alike and so they merit special recognition.

The successful completion of exhibiting all integers satisfying the requirements set forth in the problem add to further progress of Number Theory as they offer good applications in the field of Graph theory, Modular theory, Coding and Cryptography, Engineering, Music and so on. Integers have repeatedly played a crucial role in the evolution of the Natural Sciences. The theory of integers provides answers to real world problems.

The focus in this book is on solving multivariable second-degree Diophantine equations. These types of equations can be challenging as they involve finding integer solutions that satisfy the given polynomial equation. Learning about the various techniques to solve these multivariable second-degree Diophantine equations in successfully deriving their solutions help us understand how numbers work and their significance in different areas of mathematics and science. This book contains a reasonable collection of special multivariable Quadratic Diophantine problems in three, four and five variables. The process of getting different sets of integer solutions to each of the quadratic Diophantine equations considered in this book are illustrated in an elegant manner.

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Chapter 1

Designs of integer solutions to homogeneous ternary quadratic equation

1.1 Method of Analysis

The homogeneous quadratic with three unknowns is

$$x^2 + y^2 = 145z^2 \tag{1.1}$$

The process of obtaining different choices of integer solutions to (1.1) is illustrated below:

Choice 1.1

Substituting

$$x = 9\alpha + 8\beta \, , y = 8\alpha - 9\beta \tag{1.2}$$

in (1.1) , it reduces to the Pythagorean equation

$$\alpha^2 + \beta^2 = z^2 \tag{1.3}$$

which is satisfied by

$$\alpha = 2pq \, , \beta = p^2 - q^2 \, , p > q > 0 \tag{1.4}$$

and

$$z = p^2 + q^2 \tag{1.5}$$

In view of (1.2) , we have

$$x = 18pq + 8(p^2 - q^2) \, , y = 16pq - 9(p^2 - q^2) \tag{1.6}$$

Thus , (1.5) and (1.6) give the integer solutions to (1.1).

Note 1.1

However, apart from (1.2), one may consider the following transformations

$$x = 9\alpha - 8\beta\ ,\ y = 8\alpha + 9\beta\ ,$$
$$x = 8\alpha - 9\beta\ ,\ y = 9\alpha + 8\beta\ ,$$
$$x = 8\alpha + 9\beta\ ,\ y = 9\alpha - 8\beta\ ,$$
$$x = 12\alpha + \beta\ ,\ y = \alpha - 12\beta\ ,$$
$$x = 12\alpha - \beta\ ,\ y = \alpha + 12\beta\ ,$$

leading to different sets of integer solutions to (1.1).

Choice 1.2

Write (1.1) as

$$145 z^2 - x^2 = y^2 * 1 \tag{1.7}$$

Assume

$$y = 145 a^2 - b^2 \tag{1.8}$$

Write the integer 1 in (1.7) as

$$1 = (\sqrt{145} + 12)(\sqrt{145} - 12) \tag{1.9}$$

Substituting (1.8) & (1.9) in (1.7) and applying factorization, consider

$$\sqrt{145}\, z + x = (\sqrt{145} + 12)(\sqrt{145}\, a + b)^2$$

from which we have

$$\mathrm{x} = 12\,(145\mathrm{a}^2 + \mathrm{b}^2) + 290\,\mathrm{a}\,\mathrm{b}\ ,$$
$$\mathrm{z} = (145\mathrm{a}^2 + \mathrm{b}^2) + 24\,\mathrm{a}\,\mathrm{b} \tag{1.10}$$

Thus, (1.8) & (1.10) represent the integer solutions to (1.1).

Note 1.2

In addition to (1.9) , the following representations to the integer 1 in (1.7)

$$1=\frac{(\sqrt{145}+1)\,(\sqrt{145}-1)}{144}\ ,$$
$$1=\frac{(\sqrt{145}+8)\,(\sqrt{145}-8)}{81}\ ,$$
$$1=\frac{(\sqrt{145}+9)\,(\sqrt{145}-9)}{64}$$

lead to three more sets of integer solutions to (1.1).

Choice 1.3

Rewrite (1.1) in the form of ratio as

$$\frac{x+z}{144z+y}=\frac{144z-y}{x-z}=\frac{P}{Q}\ ,Q>0$$

which is equivalent to the system of double equations

$$Qx-Py+(Q-144P)z=0$$
$$Px+Qy-(P+144Q)z=0$$

Applying the method of cross-multiplication, the integer solutions to (1.1) are found to be

$$x=P^2+288PQ-Q^2\ ,$$
$$y=-144P^2+2PQ+144Q^2\ ,$$
$$z=Q^2+P^2$$

Note 1.3

Also, (1.1) may be written in the ratio forms as below:

$$\frac{x+z}{144z-y}=\frac{144z+y}{x-z}=\frac{P}{Q}\ ,Q>0,$$

$$\frac{x+9z}{8z-y}=\frac{8z+y}{x-9z}=\frac{P}{Q}\ ,Q>0,$$

$$\frac{x+9z}{8z+y}=\frac{8z-y}{x-9z}=\frac{P}{Q}\ ,Q>0.$$

Following the procedure as in Choice 1.3, different sets of solutions to (1.1) are obtained.

Choice 1.4

Assume

$$z = a^2 + b^2 \tag{1.11}$$

Write the integer 145 in (1.1) as

$$145 = (12+i)\ (12-i) \tag{1.12}$$

Substituting (1.11) & (1.12) in (1.1) and employing the method of Factorization, consider

$$x + i\,y = (12+i)\ (a + i\,b)^2$$

On equating the real and imaginary parts in the above equation , we get

$$\begin{aligned} x &= 12(a^2 - b^2) - 2\,a\,b, \\ y &= (a^2 - b^2) + 24\ a\ b. \end{aligned} \tag{1.13}$$

Thus , (1.11) & (1.13) satisfy (1.1).

Note 1.4

In addition to (1.12) , one may have the following representations to the integer 145 :

$$145 = (-12+i)\ (-12-i)\ (\pm 1+12i)\ (\pm 1-12i)\ ,(\pm 8+9i)\ (\pm 8-9i)\ ,(\pm 9+8i)\ (\pm 9-8i)$$

Following the above procedure , some more sets of integer solutions to (1.1) are obtained.

Remark 1.1

It is worth to mention that, apart from the above representation of the integer 145 as the product of complex conjugates, one may have the representation through employing the legs of Pythagorean triangle as illustrated below:

Let $p, q, p > q > 0$ denote the generators of a Pythagorean triangle. Then, the legs of the corresponding Pythagorean triangle are given by $p^2 - q^2, 2pq$. Consider two integers f, g such that

$$f = c(p^2 - q^2) + d(2pq),$$
$$g = c(2pq) - d(p^2 - q^2)$$

where

$$145 = c^2 + d^2$$

It may be observed that

$$145 = (c^2 + d^2) = \frac{(f + ig)(f - ig)}{(p^2 + q^2)^2}$$

Following the process presented in **Choice 1.4** and performing some algebra, the corresponding integer solutions to (1.1) are obtained.

Choice 1.5

Consider (1.1) as

$$x^2 + y^2 = 145z^2 * 1$$

Also , the integer 1 is written as

$$1 = \frac{(p^2 - q^2 + i2pq)(p^2 - q^2 - i2pq)}{(p^2 + q^2)^2}$$

Following the analysis similar to Choice 1.2 , the corresponding integer solutions to (1) are as below :

$$x = (p^2 + q^2)\,[(a^2 - b^2)\,(12(p^2 - q^2) - 2pq) - 2ab\,(p^2 - q^2 + 24pq)]\,,$$
$$y = (p^2 + q^2)\,[(a^2 - b^2)\,(p^2 - q^2 + 24pq) + 2ab\,(12(p^2 - q^2) - 2\,pq)]\,,$$
$$z = (p^2 + q^2)^2\,(a^2 + b^2)$$

Observation 1.1

Let (x_0, y_0, z_0) be any given solution to (1.1). Then, a formula for obtaining sequence of solutions to (1.1) based on the given solution is presented below:

$$x_n = \frac{(\alpha^n + \beta^n)}{2} x_0 + \frac{\sqrt{145}(\alpha^n - \beta^n)}{2} z_0 \ ,$$

$$z_n = \frac{(\alpha^n - \beta^n)}{2\sqrt{145}} x_0 + \frac{(\alpha^n + \beta^n)}{2} z_0 \ ,$$

$$y_n = y_0 \ , \ n = 1,2,3,...$$

where

$$\alpha = 289 + 24\sqrt{145} \ , \beta = 289 - 24\sqrt{145}$$

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Chapter 2

Patterns of integer solutions to homogeneous ternary quadratic equation

2.1 Method of Analysis

The homogeneous ternary quadratic equation to be solved is

$$x^2 - xy + 2y^2 = z^2 \tag{2.1}$$

Different processes of solving (1) are illustrated below:

Process 2.1

On completing the squares , we have

$$(2x - y)^2 + 7y^2 = 4z^2 = (2z)^2 \tag{2.2}$$

which is satisfied by

$$y = 2rs, x = \frac{7r^2 - s^2 + 2rs}{2}, z = \frac{7r^2 + s^2}{2} \tag{2.3}$$

As integer solutions are required , choose r, s to be of the same parity. That is , consider

both r, s to be even or odd.

Case 2.1 : Let $r = 2R, s = 2S$

From (2.3) , the corresponding integer solutions to (2.1) are given by

$$x = 14R^2 - 2S^2 + 4RS, y = 8RS, z = 14R^2 + 2S^2$$

Observation : 2.1 It is seen that $x > y > 0$ when

$$7R^2 - S^2 > 2RS \tag{2.4}$$

Now , taking x, y to be the generators of the Pythagorean triangle (U, V, W) with $U = 2xy, V = x^2 - y^2, W = x^2 + y^2$ we have

$$U = 16RS(14R^2 - 2S^2 + 4RS),$$
$$V = (14R^2 - 2S^2 + 12RS)\,(14R^2 - 2S^2 - 4RS),$$
$$W = (14R^2 - 2S^2 + 4RS)^2 + (8RS)^2.$$

If A, P represent the area and perimeter of the above Pythagorean triangle respectively , then it satisfies the relation

$$W - 2\frac{A}{P} \text{ is a perfect square.} \qquad (2.5)$$

Illustration :

$$R = 2, S = 3$$
$$x = 62, y = 48, z = 74$$
$$U = 96*62 = 5952, V = 62^2 - 48^2 = 1540, W = 62^2 + 48^2 = 6148$$
$$A = 4583040, P = 13640$$
$$W - 2\frac{A}{P} = 5476 = 74^2$$

In a similar manner , choosing R, S suitably satisfying (2.4) , one obtains many pythagorean triangles satisfying the relation (2.5) .

Observation :2.2

From each of the values of x, y, z, U, V, W, one may obtain second order Ramanujan numbers .

Illustration:

$$y = 48 = 1*48 = 2*24 = 3*16 = 4*12 = 6*8$$
$$= A \quad = B \quad = C \quad = D \quad = E$$

From A=B , it is seen that

$$(48+1)^2 + (24-2)^2 = (48-1)^2 + (24+2)^2$$
$$\Rightarrow 49^2 + 22^2 = 47^2 + 26^2 = 2885$$

In a similar manner, we have the following results:

$$A = C \Rightarrow 49^2 + 13^2 = 47^2 + 19^2 = 2570$$
$$A = D \Rightarrow 49^2 + 8^2 = 47^2 + 16^2 = 2465$$
$$A = E \Rightarrow 49^2 + 2^2 = 47^2 + 14^2 = 2405$$
$$B = C \Rightarrow 26^2 + 13^2 = 22^2 + 19^2 = 845$$
$$B = D \Rightarrow 26^2 + 8^2 = 22^2 + 16^2 = 740$$
$$B = E \Rightarrow 26^2 + 2^2 = 22^2 + 14^2 = 680$$
$$C = D \Rightarrow 19^2 + 8^2 = 13^2 + 16^2 = 425$$
$$C = E \Rightarrow 19^2 + 2^2 = 13^2 + 14^2 = 365$$
$$D = E \Rightarrow 16^2 + 2^2 = 8^2 + 14^2 = 260$$

Thus , 2885,2570,2465,2405,845,740,680,425,365,260 represent second order Ramanujan numbers.

Case 2.2 : Let $r = 2R + 1, s = 2S + 1$

From (2.3) , the corresponding integer solutions to (2.1) are given by

$$x = 14R^2 + 16R - 2S^2 + 4RS + 4 \,, y = 8RS + 4R + 4S + 2, z = 14R^2 + 14R + 2S^2 + 2S + 4$$

Observation : 2.3 It is seen that $x > y > 0$ when

$$(3R + 1)^2 > 2S + R^2 + (R + S)^2 \qquad (2.6)$$

Now , taking x, y to be the generators of the Pythagorean triangle (U, V, W) with $U = 2xy, V = x^2 - y^2, W = x^2 + y^2$,

we have

$$U = 2(14R^2 + 16R - 2S^2 + 4RS + 4)\ (8RS + 4R + 4S + 2)\ ,$$
$$V = (14R^2 + 16R - 2S^2 + 4RS + 4)^2 - (8RS + 4R + 4S + 2)^2,$$
$$W = (14R^2 + 16R - 2S^2 + 4RS + 4)^2 + (8RS + 4R + 4S + 2)^2 .$$

If A, P represent the area and perimeter of the above Pythagorean triangle respectively , then it satisfies the relation (2.5).

Illustration :

$$R = 2, S = 3$$
$$x = 98, y = 70, z = 112$$
$$U = 196*70 = 13720, V = 98^2 - 70^2 = 4704, W = 98^2 + 70^2 = 14504$$
$$A = 32269440, P = 32928$$
$$W - 2\frac{A}{P} = 12544 = 112^2$$

In a similar manner , choosing R, S suitably satisfying (2.6) , one obtains many pythagorean triangles satisfying the relation (2.5) .

Observation :2.4

From each of the values of x, y, z, U, V, W , one may obtain second order Ramanujan numbers .

Illustration:

$$\begin{aligned} y = 70 &= 1*70 = 2*35 = 5*14 = 7*10 \\ &= A \quad = B \quad = C \quad = D \end{aligned}$$

From A=B , it is seen that

$$(70+1)^2 + (35-2)^2 = (70-1)^2 + (35+2)^2$$
$$\Rightarrow 71^2 + 33^2 = 69^2 + 37^2 = 6130$$

In a similar manner ,we have the following results :

$$A = C \Rightarrow 71^2 + 9^2 = 69^2 + 19^2 = 5122$$
$$A = D \Rightarrow 71^2 + 3^2 = 69^2 + 17^2 = 5050$$
$$B = C \Rightarrow 37^2 + 9^2 = 33^2 + 19^2 = 1450$$
$$B = D \Rightarrow 37^2 + 3^2 = 33^2 + 17^2 = 1378$$
$$C = D \Rightarrow 19^2 + 3^2 = 9^2 + 17^2 = 370$$

Thus , 5122,5050,1450,1378,370 represent second order Ramanujan numbers.

Process 2.2

Write (2.1) in the form of ratio as

$$\frac{z-x}{y}=\frac{2y-x}{z+x}=\frac{P}{Q}, Q\neq 0 \qquad (2.7)$$

Solving the above system of double equations, we have

$$x=2Q^2-P^2, y=Q^2+2PQ, z=2Q^2+P^2+PQ$$

Observation :2.5

It is seen that $x>y>0$ when

$$2Q^2>(P+Q)^2 \qquad (2.8)$$

Now , taking x, y to be the generators of the Pythagorean triangle (U, V, W) with $U=2xy, V=x^2-y^2, W=x^2+y^2$,

we have

$$U=2(2Q^2-P^2)(Q^2+2PQ),$$
$$V=(2Q^2-P^2)^2-(Q^2+2PQ)^2,$$
$$W=(2Q^2-P^2)^2+(Q^2+2PQ)^2.$$

If A, P represent the area and perimeter of the above Pythagorean triangle respectively , then it satisfies the relation (2.5).

Illustration :

$$P=1, Q=3$$
$$x=17, y=15, z=22$$
$$U=510, V=64, W=514$$
$$A=510*32, P=1088$$
$$W-2\frac{A}{P}=484=22^2$$

In a similar manner , choosing P, Q suitably satisfying (2.8) , one obtains many pythagorean triangles satisfying the relation (2.5) .

Observation :2.6

From each of the values of x, y, z, U, V, W , one may obtain second order Ramanujan numbers .

Illustration:

$$V = 64 = 1*64 = 2*32 = 4*16$$
$$= A \quad = B \quad = C$$

From A=B , it is seen that

$$(64+1)^2 + (32-2)^2 = (64-1)^2 + (32+2)^2$$
$$\Rightarrow 65^2 + 30^2 = 63^2 + 34^2 = 5125$$

In a similar manner ,we have the following results :

$$A = C \Rightarrow 65^2 + 12^2 = 63^2 + 20^2 = 4369$$
$$B = C \Rightarrow 34^2 + 12^2 = 30^2 + 20^2 = 1300$$

Thus , 5125 ,4369 ,1300 represent second order Ramanujan numbers.

Process 2.3

Write (2.1) in the form of ratio as

$$\frac{z-x}{2y-x} = \frac{y}{z+x} = \frac{P}{Q}, Q \neq 0 \tag{2.9}$$

Solving the above system of double equations , we have

$$x = Q^2 - 2P^2, y = 2PQ - P^2, z = Q^2 + 2P^2 - PQ$$

Observation :2.7 It is seen that $x > y > 0$ when

$$Q^2 - P^2 > 2PQ \tag{2.10}$$

Now , taking x, y to be the generators of the Pythagorean triangle (U, V, W) with $U = 2xy, V = x^2 - y^2, W = x^2 + y^2$, we have

$$U = 2(Q^2 - 2P^2)(2PQ - P^2),$$
$$V = (Q^2 - 2P^2)^2 - (2PQ - P^2)^2,$$
$$W = (Q^2 - 2P^2)^2 + (2PQ - P^2)^2.$$

If A, P represent the area and perimeter of the above Pythagorean triangle respectively , then it satisfies the relation (2.5).

Illustration :

$$P = 1, Q = 3$$
$$x = 7, y = 5, z = 8$$
$$U = 70, V = 24, W = 74$$
$$A = 840, P = 168$$
$$W - 2\frac{A}{P} = 64 = 8^2$$

In a similar manner , choosing P, Q suitably satisfying (2.10) , one obtains many pythagorean triangles satisfying the relation (2.5) .

Observation :2.8

From each of the values of x, y, z, U, V, W, one may obtain second order Ramanujan numbers .

Illustration:

$$V = 24 = 1*24 = 2*12 = 3*8 = 4*6$$
$$= A \quad = B \quad = C \quad = D$$

From A=B , it is seen that

$$(24+1)^2 + (12-2)^2 = (24-1)^2 + (12+2)^2$$
$$\Rightarrow 25^2 + 10^2 = 23^2 + 14^2 = 725$$

In a similar manner ,we have the following results :

$$A = C \Rightarrow 25^2 + 5^2 = 23^2 + 11^2 = 650$$
$$A = D \Rightarrow 25^2 + 2^2 = 23^2 + 10^2 = 629$$
$$B = C \Rightarrow 14^2 + 5^2 = 10^2 + 11^2 = 221$$
$$C = D \Rightarrow 11^2 + 2^2 = 5^2 + 10^2 = 125$$

Thus , 725 ,650 ,629 ,221,125 represent second order Ramanujan numbers.

Process 2.4

Write (2.2) as

$$4z^2 - 7y^2 = (2x - y)^2 \tag{2.11}$$

Assume

$$2x - y = 4(a^2 - 7b^2) \tag{2.12}$$

Substituting (2.12) in (2.11) and applying factorization , we have

$$2z+\sqrt{7}\,y=4(a+\sqrt{7}\,b)^2=4a^2+28b^2+8\sqrt{7}\,ab$$

Equating the coefficients of corresponding terms in the above equation , we get

$$z=2a^2+14b^2\,,\,y=8ab \qquad (2.13)$$

In view of (2.12) , we have

$$x=2a^2-14\,b^2+4ab \qquad (2.14)$$

Observation :2.9 It is seen that $x>y>0$ when

$$(a-b)^2>8b^2 \qquad (2.15)$$

Now , taking $x\,,y$ to be the generators of the Pythagorean triangle $(U\,,V\,,W)$ with $U=2xy, V=x^2-y^2\,,W=x^2+y^2$,

we have

$$U=16ab(2a^2-14b^2+4ab),$$
$$V=(2a^2-14b^2+4ab)^2-(8ab)^2\,,$$
$$W=(2a^2-14b^2+4ab)^2+(8ab)^2\,.$$

If $A\,,P$ represent the area and perimeter of the above Pythagorean trianglerespectively , then it satisfies the relation (2.5).

Illustration :

$$a=4\,,b=1$$
$$x=34, y=32\,,z=46$$
$$U=2176, V=132\;, W=2180$$
$$A=143616\,,P=4488$$
$$W-2\frac{A}{P}=\;2116=46^2$$

In a similar manner , choosing $a\,,b$ suitably satisfying (2.15) , one obtains many pythagorean triangles satisfying the relation (2.5) .

Observation :2.10

From each of the values of x, y, z, U, V, W, one may obtain second order Ramanujan numbers .

Illustration:

$$y = 32 = 1*32 = 2*16 = 4*8$$
$$= A \quad = B \quad = C$$

From A=B , it is seen that

$$(32+1)^2 + (16-2)^2 = (32-1)^2 + (16+2)^2$$
$$\Rightarrow 33^2 + 14^2 = 31^2 + 18^2 = 1285$$

In a similar manner ,we have the following results :

$$A = C \Rightarrow 33^2 + 4^2 = 31^2 + 12^2 = 1105$$
$$B = C \Rightarrow 18^2 + 4^2 = 14^2 + 12^2 = 340$$

Thus , 1285 ,1105 ,340 represent second order Ramanujan numbers.

Process 5

Treating (2.1) as a quadratic in x and solving for the same ,we get

$$x = \frac{y \pm \sqrt{4z^2 - 7y^2}}{2} \qquad (2.16)$$

It is possible to choose y ,z so that the square-root in (2.16) is eliminated and obtain the corresponding value to x satisfying (2.1). For simplicity and brevity ,a few examples are presented as below :

Example

$$y = 2s + 1, z = s^2 + s + 2, x = s^2 + 2s - 1$$

It is seen that $x > y > 0$ when

$$(a-b)^2 > 8b^2 \qquad (2.17)$$

Now , taking x, y to be the generators of the Pythagorean triangle (U, V, W) with $U = 2xy, V = x^2 - y^2, W = x^2 + y^2$,

we have

$$U = 16ab(2a^2 - 14b^2 + 4ab),$$
$$V = (2a^2 - 14b^2 + 4ab)^2 - (8ab)^2,$$
$$W = (2a^2 - 14b^2 + 4ab)^2 + (8ab)^2.$$

If A, P represent the area and perimeter of the above Pythagorean triangle respectively , then it satisfies the relation (2.5).

Illustration :

$$a = 4, b = 1$$
$$x = 34, y = 32, z = 46$$
$$U = 2176, V = 132, W = 2180$$
$$A = 143616, P = 4488$$
$$W - 2\frac{A}{P} = 2116 = 46^2$$

In a similar manner , choosing a, b suitably satisfying (2.15) , one obtains many pythagorean triangles satisfying the relation (2.5) .

Observation :2. 10

From each of the values of x, y, z, U, V, W, one may obtain second order Ramanujan numbers .

Illustration:

$$y = 32 = 1*32 = 2*16 = 4*8$$
$$= A \quad = B \quad = C$$

From A=B , it is seen that

$$(32+1)^2 + (16-2)^2 = (32-1)^2 + (16+2)^2$$
$$\Rightarrow 33^2 + 14^2 = 31^2 + 18^2 = 1285$$

In a similar manner ,we have the following results :

$$A = C \Rightarrow 33^2 + 4^2 = 31^2 + 12^2 = 1105$$
$$B = C \Rightarrow 18^2 + 4^2 = 14^2 + 12^2 = 340$$

Thus , 1285 ,1105 ,340 represent second order Ramanujan numbers.

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Chapter 3

A scrutiny of integer solutions to non-homogeneous ternary quadratic equation

3.1 Method of Analysis:

Consider the diophantine equation representing hyperboloid of one sheet given by

$$x^2+2y^2-z^2=2 \tag{3.1}$$

The process of obtaining patterns of integer solutions to (1) is illustrated below:

Pattern I

Assuming

$$x=ky\,, k>1 \tag{3.2}$$

in (3.1) ,it is written as

$$z^2=\left(k^2+2\right)y^2-2 \tag{3.3}$$

with the least positive integer solution

$$y_0=1, z_0=k$$

To obtain the other solutions of (3.3) ,consider the pellian equation

$$z^2=\left(k^2+2\right)y^2+1$$

whose general solution $(\tilde{y}_n\,,\tilde{z}_n)$ is given by

$$\tilde{z}_n=\frac{f_{n,k}}{2}\,,\tilde{y}_n=\frac{g_{n,k}}{2\sqrt{k^2+2}} \text{ in which}$$

$$f_{n,k}=(k^2+1+k\sqrt{k^2+2})^{n+1}+(k^2+1-k\sqrt{k^2+2})^{n+1},$$

$$g_{n,k}=(k^2+1+k\sqrt{k^2+2})^{n+1}-(k^2+1-k\sqrt{k^2+2})^{n+1}$$

Applying Brahmagupta lemma between the solutions of $(y_0\,,z_0)$ and $(\tilde{y}_n\,,\tilde{z}_n)$,the general solution of (3) is found to be

$$\left.\begin{aligned} y_{n+1}&=\frac{f_{n,k}}{2}+\frac{k}{2\sqrt{k^2+2}}g_{n,k}\ , \\ z_{n+1}&=k\frac{f_{n,k}}{2}+\frac{\sqrt{k^2+2}}{2}g_{n,k}\ , n=-1,0,1,2, \end{aligned}\right\} \tag{3.4}$$

In view of (3.2) ,we have

$$x_{n+1} = k[\frac{f_{n,k}}{2} + \frac{k}{2\sqrt{k^2+2}} g_{n,k}] \tag{3.5}$$

Thus,(3.4) and (3.5) represent the integer solutions to (3.1).
A few examples are given in Table 3.1 below:

Table 3.1-Examples

n	x_{n+1}	y_{n+1}	z_{n+1}
-1	k	1	k
0	$k(2k^2+1)$	$(2k^2+1)$	$(2k^3+3k)$
1	$k(4k^4+6k^2+1)$	$(4k^4+6k^2+1)$	$(4k^5+10k^3+5k)$

The recurrence relations satisfied by the values of x, y, z are respectively given by

$x_{n+3} - 2(k^2+1)x_{n+2} + x_{n+1} = 0$,

$y_{n+3} - 2(k^2+1)y_{n+2} + y_{n+1} = 0$,

$z_{n+3} - 2(k^2+1)z_{n+2} + z_{n+1} = 0$.

A few interesting properties are given below:

(i) $6[(k^2+2)y_{2n+2} - kz_{2n+2} + 2]$ is a square multiple of 6

(ii) $6[(k^2+2)y_{2n+2} - kz_{2n+2} + 2]$ is a cubic integer.

(iii) $[(k^2+2)y_{4n+4} - kz_{4n+4} + 2]$ is a perfect square.

(iv) $[(k^2+2)y_{n+1} - kz_{n+1}]^2 - (k^2+2)[ky_{n+1} - z_{n+1}]^2 = 4$

Pattern II

Substitution of

$$z = ky, (k > 1) \tag{3.6}$$

in (3.1) leads to

$$x^2 = (k^2-2)y^2 + 2 \tag{3.7}$$

with the least positive integer solution

$y_0 = 1, x_0 = k$

To obtain the other solutions of (3.7) , consider the pellian equation

$$x^2 = (k^2-2)y^2 + 1$$

whose general solution $(\tilde{y}_n, \tilde{x}_n)$ is given by

$$\tilde{x}_n = \frac{f_{n,k}}{2}, \tilde{y}_n = \frac{g_{n,k}}{2\sqrt{k^2-2}} \text{ in which}$$

$$f_{n,k} = (k^2 - 1 + k\sqrt{k^2-2})^{n+1} + (k^2 - 1 - k\sqrt{k^2-2})^{n+1},$$

$$g_{n,k} = (k^2 - 1 + k\sqrt{k^2-2})^{n+1} - (k^2 - 1 - k\sqrt{k^2-2})^{n+1}$$

Applying Brahmagupta lemma between the solutions of (y_0, x_0) and $(\tilde{y}_n, \tilde{x}_n)$, the general solution of (3.7) is found to be

$$\left.\begin{aligned} y_{n+1} &= \frac{f_{n,k}}{2} + \frac{k}{2\sqrt{k^2-2}} g_{n,k}, \\ x_{n+1} &= k\frac{f_{n,k}}{2} + \frac{\sqrt{k^2-2}}{2} g_{n,k}, n = -1,0,1,2, \end{aligned}\right\} \quad (3.8)$$

In view of (3.6) ,we have

$$z_{n+1} = k\left[\frac{f_{n,k}}{2} + \frac{k}{2\sqrt{k^2-2}} g_{n,k}\right] \quad (3.9)$$

Thus,(3.8) and (3.9) represent the integer solutions to (3.1).
A few examples are given in Table 3.2 below:

Table 3.2-Examples

n	x_{n+1}	y_{n+1}	z_{n+1}
-1	k	1	k
0	$(2k^3 - 3k)$	$(2k^2 - 1)$	$(2k^3 - k)$
1	$k(4k^4 - 10k^2 + 5)$	$(4k^4 - 6k^2 + 1)$	$(4k^5 - 6k^3 + k)$

The recurrence relations satisfied by the values of x, y, z are respectively given by

$$x_{n+3} - 2(k^2 - 1)x_{n+2} + x_{n+1} = 0,$$

$$y_{n+3} - 2(k^2 - 1)y_{n+2} + y_{n+1} = 0,$$

$$z_{n+3} - 2(k^2 - 1)z_{n+2} + z_{n+1} = 0.$$

A few interesting properties are given below:

(v) $6[kx_{2n+2} - (k^2 - 2)y_{2n+2} + 2]$ is a square multiple of 6

(vi) $kx_{3n+3} - (k^2 - 2)y_{3n+3} + 3(kx_{n+1} - (k^2 - 2)y_{n+1})$ is a cubic integer.

(vii) $kx_{4n+4} - (k^2 - 2)y_{4n+4} + 2$ is a perfect square.

(viii) $[\, k x_{n+1} - (k^2-2) y_{n+1}]^2 - (k^2-2)[k y_{n+1} - x_{n+1}]^2 = 4$

Pattern III

Substitution of

$$z = u+v \,, x = u-v \,, u \neq v \neq 0 \qquad (3.10)$$

in (3.1) leads to

$$y^2 = 2uv+1 \qquad (3.11)$$

It is possible to choose u, v such that the R.H.S. of (3.11) is a perfect square and taking its square-root ,the value of y is obtained. Substituting the above values of u, v in (3.10) ,the corresponding values of x, z satisfying (3.1) are found. A few examples are given in Table 3.3 below :

Table 3.3-examples

u	v	x	y	z
$2\,s$	$s+1$	$s-1$	$2s+1$	$3s+1$
$2\,s$	$s-1$	$s+1$	$2s-1$	$3s-1$
$s(s+1)$	2	s^2+s-2	$2s+1$	s^2+s+2
$6*2^n$	$3*2^n-1$	$3*2^n+1$	$6*2^n-1$	$9*2^n-1$

Pattern IV

Taking

$$z = (2s+3)\; x \qquad (3.12)$$

in (3.1) ,we have the well-known pellian equation

$$y^2 = (2s^2+6s+4)\; x^2+1 \qquad (3.13)$$

If (x_0 , y_0) is the initial solution to (3.13) ,then its general solution (x_{n+1}, y_{n+1}) is given by

$$y_{n+1} = \frac{f_n}{2} \,, x_{n+1} = \frac{g_n}{2\sqrt{2s^2+6s+4}} \qquad (3.14)$$

where

$$f_n = (y_0 + \sqrt{2s^2+6s+4}\, x_0)^{n+1} + (y_0 - \sqrt{2s^2+6s+4}\, x_0)^{n+1} \,,$$

$$g_n = (y_0 + \sqrt{2s^2+6s+4}\, x_0)^{n+1} - (y_0 - \sqrt{2s^2+6s+4}\, x_0)^{n+1} \,.$$

In view of (12) , we have

$$z_{n+1} = (2s+3)\,\frac{g_n}{2\sqrt{2s^2+6s+4}} \tag{3.15}$$

Thus, (3.14) and (3.15) represent the integer solutions to (3.1).
The recurrence relations satisfied by the values of x, y, z are respectively given by

$$x_{n+3} - 2\,y_0\,x_{n+2} + x_{n+1} = 0\,,$$
$$y_{n+3} - 2\,y_0\,y_{n+2} + y_{n+1} = 0\,,$$
$$z_{n+3} - 2\,y_0\,z_{n+2} + z_{n+1} = 0\,.$$

Remarkable Observation

Let (x_0, y_0, z_0) be any given integer solution to (1). Then ,the triple (x_n, y_n, z_n) Given by

$$x_n = x_0\,,$$
$$y_n = Y_{n-1}\,y_0 + Z_{n-1}\,z_0\,,$$
$$z_n = 2\,Z_{n-1}\,y_0 + Y_{n-1}\,z_0\,, n = 1,2,3,...$$

also satisfy (3.1) where

(Y_{n-1}, Z_{n-1}) being the solution of the pellian equation $Y^2 = 2Z^2 + 1$.

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Chapter 4

A portrayal of integer solutions to homogeneous ternary quadratic equation

4.1 Method of Analysis:

The quadratic Diophantine equation with three unknowns studied for its non-zero distinct integer solutions is given by

$$x^2 = 25y^2 + 29z^2 \tag{4.1}$$

We illustrate below different sets of integral solutions of (4.1).

Set I

It is observed that (4.1) is of the form

$$x^2 = y^2 + Dz^2 \tag{4.2}$$

where $D = 29$. Employing the most cited solutions of (4.2), one may obtain

$$x = 29m^2 + n^2$$
$$y = \frac{1}{5}\left(29m^2 - n^2\right)$$
$$z = 2mn, m, n \in N.$$

Since our interest centers on finding integral solutions, it is possible to choose m, n such that x, y and z are integers. For the sake of clear understanding, the values of m, n with the corresponding solutions are presented in Table 4.1 below:

Table 4.1: Values of *m*, *n* with solutions

Choices	*m*	*N*	*x, y, z*
1	$5M$	$5N$	$725M^2+25N^2, 145M^2-5N^2, 50MN$
2	$5k-4$	$5k-3$	$750k^2-1190k+473, 140k^2-226k+91, 50k^2-70k+24$
3	$5k-4$	$5k-2$	$750k^2-1180k+468, 140k^2-228k+92, 50k^2-60k+16$
4	$5k-3$	$5k-4$	$750k^2-910k+277, 140k^2-166k+49, 50k^2-70k+24$
5	$5k-3$	$5k-1$	$750k^2-880k+262, 140k^2-172k+52, 50k^2-40k+6$
6	$5k-2$	$5k-4$	$750k^2-620k+132, 140k^2-108k+20, 50k^2-60k+16$
7	$5k-2$	$5k-1$	$750k^2-590k+117, 140k^2-114k+23, 50k^2-30k+4$
8	$5k-1$	$5k-3$	$750k^2-320k+38, 140k^2-52k+4, 50k^2-40k+6$
9	$5k-1$	$5k-2$	$750k^2-310k+33, 140k^2-54k+5, 50k^2-30k+4$

Set II

Express (4.1) as the system of double equations as presented in Table 4.2 below:

Table 4.2: System of double equations

System	I	II	III
$x+5y$	z^2	$29z^2$	$29z$
$x-5y$	29	1	z

Solving each of the above system of double equations, one obtains the corresponding integer solutions to (4.1) as exhibited below:

Solutions to System I

$$x=50k^2+30k+19$$
$$y=10k^2+6k-2$$
$$z=10k+3$$

Solutions to System II

$$x = 1450k^2 + 870k + 131$$
$$y = 290k^2 + 174k + 26$$
$$z = 10k + 3$$

Solutions to System III

$$x = 75\alpha, y = 14\alpha, z = 5\alpha$$

Set III

Write (4.1) as

$$25y^2 + 29z^2 = x^2 * 1 \tag{4.3}$$

Let $x = 25a^2 + 29b^2$ (4.4)

Write 1 on the right hand side of (4.3) as

$$1 = \frac{\left(14 + i\sqrt{29}\right)\left(14 - i\sqrt{29}\right)}{15^2} \tag{4.5}$$

Substituting (4.4) and (4.5) in (4.3) and employing the factorization method,

define

$$5y + i\sqrt{29}z = \frac{1}{15}\left(5a + i\sqrt{29}b\right)^2 (14 + i\sqrt{29})$$

Equating real and imaginary parts, we've

$$\left.\begin{aligned} 5y &= \frac{1}{15}\left[350a^2 - 406b^2 - 290ab\right] \\ z &= \frac{1}{5}\left[25a^2 - 29b^2 + 140ab\right] \end{aligned}\right\} \tag{4.6}$$

As our interest is finding integer solutions, we choose a and b suitably so that x, y, z are integers,

Replacing a by 15a and b by 15b in (4.6) and (4.4), the corresponding integer solutions to (4.1) are given by

$$x = x(a,b) = 5625a^2 + 6525b^2$$
$$y = y(a,b) = 1050a^2 - 1218b^2 - 870ab$$
$$z = z(a,b) = 375a^2 - 435b^2 + 2100ab$$

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Chapter 5

Designs of integer solutions to homogeneous ternary quadratic equation

5.1 Method of Analysis

The ternary quadratic equation to be solved for its integer solutions is

$$z^2 = \left(2k^2 - 2k + 22\right)x^2 + y^2 \tag{5.1}$$

We present below different methods of solving (5.1):

Method: 1

(5.1) is written in the form of ratio as

$$\frac{z+y}{\left(2k^2 - 2k + 22\right)x} = \frac{x}{z-y} = \frac{r}{s}, s \neq 0$$

which is equivalent to the system of double equations

$$\left(2k^2 - 2k + 22\right)rx - sy - sz = 0$$
$$sx + ry - rz = 0$$

Applying the method of cross-multiplication to the above system of equations,

$$x = x(r,s) = 2rs$$
$$y = y(r,s) = \left(2k^2 - 2k + 22\right)r^2 - s^2$$
$$z = z(r,s) = \left(2k^2 - 2k + 22\right)r^2 + s^2$$

which satisfy (5.1)

Note: 1

It is observed that (5.1) may also be represented in the form of ratio as below:

(i) $\frac{z+y}{2x}=\frac{\left(k^2-k+11\right)x}{z-y}=\frac{r}{s}, s\neq 0$

The corresponding solutions to (5.1) are given as:

$$x=2rs, y=2r^2-\left(k^2-k+11\right)s^2, z=2r^2+\left(k^2-k+11\right)s^2$$

(ii) $\frac{z+y}{\left(k^2-k+11\right)x}=\frac{2x}{z-y}=\frac{r}{s}, s\neq 0$

The corresponding solutions to (5.1) are given as:

$$x=2rs, y=\left(k^2-k+11\right)r^2-2s^2, z=\left(k^2-k+11\right)r^2+2s^2$$

Method: 2

(5.1) is written as the system of double equation in Table 5.1 as follows:

Table 5.1: System of Double Equations

System	1	2	3	4
$z+y$	$2x$	$\left(k^2-k+11\right)x^2$	$\left(2k^2-2k+22\right)x$	$\left(k^2-k+11\right)x$
$z-y$	$\left(k^2-k+11\right)x$	2	x	$2x$

Solving each of the above system of double equations, the value of x, y & z satisfying (5.1) are obtained. For simplicity and brevity, in what follows, the integer solutions thus obtained are exhibited.

Solutions for system: I

$$x = 2s, y = -\left(k^2-k+9\right)s, z = \left(k^2-k+13\right)s$$

Solutions for system: II

$$x=2s,\ y=2s^2\left(k^2-k+11\right)-1,\ z=2s^2\left(k^2-k+11\right)+1$$

Solution for system: III

$$x=2s,\ y=\left(2k^2-2k+21\right)s,\ z=\left(2k^2-2k+23\right)s$$

Solution for system: IV

$$x=2s,\ y=s\left(k^2-k+11\right)-2s,\ z=s\left(k^2-k+11\right)+2s$$

Method: 3

(5.1) is written as

$$y^2+\left(2k^2-2k+22\right)x^2=z^2=z^2*1 \tag{5.2}$$

Assume z as

$$z=a^2+\left(2k^2-2k+22\right)b^2 \tag{5.3}$$

Write 1 as

$$1=\frac{\left[\left(2k^2-2k+22\right)r^2-s^2+i2rs\sqrt{2k^2-2k+22}\right]*\left[\left(2k^2-2k+22\right)r^2-s^2-i2rs\sqrt{2k^2-2k+22}\right]}{\left(\left(2k^2-2k+22\right)r^2+s^2\right)^2} \tag{5.4}$$

Using (5.3) & (5.4) in (5.2) and employing the method of factorization, consider

$$y+i\sqrt{2k^2-2k+22}\,x=\frac{\left(a+ib\sqrt{2k^2-2k+22}\right)^2\left[\left(2k^2-2k+22\right)r^2-s^2+i\sqrt{2k^2-2k+22}\ 2rs\right]}{\left(2k^2-2k+22\right)r^2+s^2}$$

Equating real & imaginary parts, it is seen that

$$\left.\begin{aligned} y&=\frac{1}{\left(2k^2-2k+22\right)r^2+s^2}\left[\left\{\left(2k^2-2k+22\right)r^2-s^2\right\}\left\{a^2-\left(2k^2-2k+22\right)b^2\right\}-4abrs\left\{2k^2-2k+22\right\}\right] \\ x&=\frac{1}{\left(2k^2-2k+22\right)r^2+s^2}\left[2ab\left\{\left(2k^2-2k+22\right)r^2-s^2\right\}+2rs\left\{a^2-\left(2k^2-2k+22\right)b^2\right\}\right] \end{aligned}\right\}$$

(5.5)

Since our interest is to find the integer solutions, replacing a by $\left[\left(2k^2-2k+22\right)r^2+s^2\right]A$ & b by $\left[\left(2k^2-2k+22\right)r^2+s^2\right]B$ in (5.5) & (5.3), the corresponding integer solutions to (5.1) are given by

$$x=x(A,B)=\left(\left(2k^2-2k+22\right)r^2+s^2\right)\left[\left(\mathrm{A}^2-\left(2k^2-2k+22\right)B^2\right)2rs+2AB\left(\left(2k^2-2k+22\right)r^2-s^2\right)\right]$$

$$y=y\left(\mathrm{A},\mathrm{B}\right)=\left(\left(2k^2-2k+22\right)r^2+s^2\right)\left[\begin{array}{r}\left(\mathrm{A}^2-\left(2k^2-2k+22\right)\mathrm{B}^2\right)\left[\left(2k^2-2k+22\right)r^2-s^2\right] \\ -4ABrs\left(2k^2-2k+22\right)\end{array}\right]$$

$$z=z\left(\mathrm{A},\mathrm{B}\right)=\left(\left(2k^2-2k+22\right)r^2+s^2\right)^2\left(\mathrm{A}^2+\left(2k^2-2k+22\right)\mathrm{B}^2\right)$$

Method: 4

(5.1) is written as

$$z^2-\left(2k^2-2k+22\right)x^2=y^2=y^2*1 \qquad (5.6)$$

Assume y as

$$y=a^2-\left(2k^2-2k+22\right)b^2 \qquad (5.7)$$

Write 1 as

$$1=\frac{\left(\left(2k^2-2k+22\right)r^2+s^2+\sqrt{2k^2-2k+22}\,2rs\right)\left(\left(2k^2-2k+22\right)r^2+s^2-\sqrt{2k^2-2k+22}\,2rs\right)}{\left(\left(2k^2-2k+22\right)r^2-s^2\right)^2}$$

(5.8)

Using (5.7) & (5.8) in (5.6) and employing the method of factorization, consider

$$z+\sqrt{2k^2-2k+22}\,x=\frac{\left[\left(2k^2-2k+22\right)r^2+s^2+2rs\sqrt{2k^2-2k+22}\right]*\begin{bmatrix}a^2+\left(2k^2-2k+22\right)b^2\\+2ab\sqrt{2k^2-2k+22}\end{bmatrix}}{\left(2k^2-2k+22\right)r^2-s^2}$$

Equating rational and irrational parts, it is seen that,

$$\left.\begin{aligned}x&=\frac{\left(a^2+\left(2k^2-2k+22\right)b^2\right)2rs+2ab\left(\left(2k^2-2k+22\right)r^2+s^2\right)}{\left(2k^2-2k+22\right)r^2-s^2}\\z&=\frac{\left(a^2+\left(2k^2-2k+22\right)b^2\right)\left(\left(2k^2-2k+22\right)r^2+s^2\right)+4abrs\left(2k^2-2k+22\right)}{\left(2k^2-2k+22\right)r^2-s^2}\end{aligned}\right\}$$

(5.9)

Since our interest to find the integer solution, replacing a by $\left(\left(2k^2-2k+22\right)r^2-s^2\right)A$ & b by $\left(\left(2k^2-2k+22\right)r^2-s^2\right)B$ in (5.7) & (5.9), the corresponding integer solutions to (5.1) are given by

$$x = x(A,B) = \left(\left(2k^2-2k+22\right)r^2-s^2\right)\left[\left(\mathrm{A}^2+\left(2k^2-2k+22\right)\mathrm{B}^2\right)2rs+2\mathrm{AB}\left(\left(2k^2-2k+22\right)r^2+s^2\right)\right]$$

$$y = y(A,B) = \left(\left(2k^2-2k+22\right)r^2-s^2\right)^2\left[\mathrm{A}^2-\left(2k^2-2k+22\right)\mathrm{B}^2\right]$$

$$z = z(A,B) = \left(\left(2k^2-2k+22\right)r^2-s^2\right)\begin{bmatrix}\left(\mathrm{A}^2+\left(2k^2-2k+22\right)\mathrm{B}^2\right)\left(\left(2k^2-2k+22\right)r^2+s^2\right)\\ +4ABrs\left(2k^2-2k+22\right)\end{bmatrix}$$

GENERATION OF SOLUTIONS

Different formulas for generating sequence of integer solutions based on the given solution are presented below:

Let $\left(x_0, y_{0,} z_0\right)$ be any given solution to (5.1).

Formula: 1

Let $\left(x_{1,} y_1, z_1\right)$ given by

$$x_1 = 3x_0, y_1 = 3y_0 + h, z_1 = 3z_0 + 2h \tag{5.10}$$

be the 2nd solution to (5.1). Using (5.10) in (5.1) and simplifying, one obtains

$$h = 2y_0 - 4z_0$$

In view of (5.10), the values of y_1 and z_1 are written in the matrix form as

$$\left(y_1, z_1\right)^t = M\left(y_0, z_0\right)^t$$

where $M = \begin{bmatrix} 5 & -4 \\ 4 & -5 \end{bmatrix}$

and t is the transpose

The repetition of the above proses leads to the n^{th} solutions y_n, z_n given by

$$\left(y_n, z_n\right)^t = M^n \left(y_0, z_0\right)^t$$

If α, β are the distinct eigen values of *M*, then

$$\alpha = 3, \beta = -3$$

We know that

$$M^n = \frac{a^n}{(\alpha-\beta)}\left(M - \beta I\right) + \frac{\beta^n}{(\beta-\alpha)}\left(M - \alpha I\right),\ I = 2\times 2 \text{ Identity matrix}$$

Thus, the general formulas for integer solutions to (5.1) are given by

$$x_n = 3^n x_0$$

$$\begin{pmatrix} y_n \\ z_n \end{pmatrix} = \frac{1}{3}\begin{bmatrix} 4\alpha^n - \beta^n & -2\alpha^n + 2\beta^n \\ 2\alpha^n - 2\beta^n & -\alpha^n + 4\beta^n \end{bmatrix}\begin{bmatrix} y_0 \\ z_0 \end{bmatrix}$$

Formula: 2

Let $\left(x_1, y_1, z_1\right)$ given by

$$x_1 = h - \left(2k^2 - 2k + 23\right)x_0,\ y_1 = h - \left(2k^2 - 2k + 23\right)y_0,\ z_1 = \left(2k^2 - 2k + 23\right)z_0$$
(5.11)

be the 2nd solution to (5.1). Using (5.11) in (5.1) and simplifying, one obtains

$$h = \left(4k^2 - 4k + 44\right)x_0 + 2y_0$$

In view of (5.11), the values of x_1 and y_1 are written in the matrix form as

$$\left(x_1, y_1\right)^t = M\left(x_0, y_0\right)^t$$

where $M = \begin{bmatrix} 2k^2 - 2k + 21 & 2 \\ 4k^2 - 4k + 44 & -(2k^2 - 2k + 21) \end{bmatrix}$

And *t* is the transpose

The repetition of the above process leads to the n^{th} solutions x_n, y_n given by

$$\left(x_n, y_n\right)^t = M^n \left(x_o, y_0\right)^t$$

If α, β are the distinct eigen values of *M*, then

$$\alpha = 2k^2 - 2k + 23 \text{ , } \beta = -\left(2k^2 - 2k + 23\right)$$

Thus, the general formulas for integer solutions to (2.113) are given by

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \frac{1}{\left(2k^2 - 2k + 23\right)} \begin{bmatrix} \left(2k^2 - 2k + 22\right)\alpha^n + \beta^n & \alpha^n - \beta^n \\ \left(2k^2 - 2k + 22\right)\left(\alpha^n - \beta^n\right) & \alpha^n + \left(2k^2 - 2k + 22\right)\beta^n \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

$$z_n = \left(2k^2 - 2k + 23\right)^n z_0$$

Formula: 3

Let $\left(x_1, y_1 z_1\right)$ given by

$$x_1 = h - \left(2k^2 - 2k + 21\right)x_0 \text{ , } y_1 = \left(2k^2 - 2k + 21\right)y_0,\ z_1 = \left(2k^2 - 2k + 21\right)z_0 + h$$
(5.12)

be the 2^{nd}solution to (5.1). Using (5.12) in (5.1) and simplifying, one obtains

$$h = 2z_0 + \left(4k^2 - 4k + 44\right)x_0$$

In view of (5.12), the values of x_1 and z_1 are written in the matrix form as

$$\left(x_1, z_1\right)^t = M\left(x_0, z_0\right)^t$$

where $M = \begin{bmatrix} 2k^2-2k+23 & 2 \\ 4k^2-4k+44 & 2k^2-2k+23 \end{bmatrix}$

and t is the transpose

The repetition of the above process leads to the n[th] solutions x_n, z_n given by

$$(x_n, z_n)^t = M^n (x_0, z_0)^t$$

If α, β are the distinct eigen values of *M*, then

$$\alpha = 2k^2-2k+23+2\sqrt{2k^2-2k+22},$$
$$\beta = 2k^2-2k+23-2\sqrt{2k^2-2k+22}$$

Thus, the general formulas for integer solutions to (5.1) are given by

$$y_n = \left(2k^2-2k+21\right)^n y_0$$

$$\begin{pmatrix} x_n \\ z_n \end{pmatrix} = \frac{1}{2}\begin{bmatrix} \alpha^n+\beta^n & \dfrac{\alpha^n-\beta^n}{2\sqrt{2k^2-2k+22}} \\ \sqrt{2k^2-2k+22}(\alpha^n-\beta^n) & \alpha^n+\beta^n \end{bmatrix}\begin{bmatrix} x_0 \\ z_0 \end{bmatrix}$$

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Chapter 6

Patterns of integer solutions to homogeneous quaternary quadratic equation

6.1 Method of Analysis:

The polynomial equation of second degree with four unknowns to be solved is

$$x^2 - 6y^2 + 15z^2 = w^2 \tag{6.1}$$

The procedure to obtain various patterns of integer solutions to (6.1) is as below:

Procedure 1

The option

$$w = 4z \tag{6.2}$$

in (6.1) gives

$$x^2 = 6y^2 + z^2 \tag{6.3}$$

which is satisfied by

$$y = 2rs, z = 6r^2 - s^2, x = 6r^2 + s^2 \tag{6.4}$$

From (6.2) , we get

$$w = 4(6r^2 - s^2) \tag{6.5}$$

Thus, (6.4) & (6.5) satisfy (6.1).

Note 6.1

It is seen that ,by expressing (6.3) as the system of double equations ,the following four patterns of integer solutions to (6.1) are obtained:

Pattern 1

$$x = 5s, y = 2s, z = s, w = 4s$$

Pattern 2

$$x = 7s, y = 2s, z = 5s, w = 20s$$

Pattern 3

$$x = 2s^2 + 3, y = 2s, z = 2s^2 - 3, w = 4(2s^2 - 3)$$

Pattern 4

$$x = 6s^2 + 1, y = 2s, z = 6s^2 - 1, w = 4(6s^2 - 1)$$

Note 6.2

Rewrite (6.3) as

$$z^2 + 6y^2 = x^2 * 1 \tag{6.6}$$

Assume

$$x = 25(a^2 + 6b^2) \tag{6.7}$$

Express the integer 1 in (6.6) as

$$1 = \frac{(1 + i2\sqrt{6})(1 - i2\sqrt{6})}{25} \tag{6.8}$$

Substituting (6.7) & (6.8) in (6.6) and using factorization, we have

$$z + i\sqrt{6}\,y = 5(1 + i2\sqrt{6})(a + i\sqrt{6}\,b)^2$$

from which, we get

$$\begin{aligned} z &= 5(a^2 - 6b^2) - 120ab, \\ y &= 10(a^2 - 6b^2) + 10ab. \end{aligned} \tag{6.9}$$

In view of (6.2) ,observe that

$$w = 20(a^2 - 6b^2) - 480ab \tag{6.10}$$

Thus,(6.7),(6.9) and (6.10) satisfy (6.1).

Remark 6.1

It is to be noted that the integer 1 in (6.6) may be considered as

$$1=\frac{(6r^2-s^2+i2rs\sqrt{6})(6r^2-s^2-i2rs\sqrt{6})}{(6r^2+s^2)^2}$$

Repeating the above process and taking different values to r& s, one obtains different sets of integer solutions to (6.1).

Procedure 2

Consider (6.3) as

$$x^2-6y^2=z^2*1 \tag{6.11}$$

Assume

$$z=a^2-6b^2 \tag{6.12}$$

The integer 1 in (6.11) is written as

$$1=(5+2\sqrt{6})(5-2\sqrt{6}) \tag{6.13}$$

Substituting (6.12) & (6.13) in (6.11) and using factorization, we have

$$x+\sqrt{6}\,y=(5+2\sqrt{6})(a+\sqrt{6}\,b)^2$$

from which, we get

$$\begin{aligned} x&=5(a^2+6b^2)+24ab,\\ y&=2(a^2+6b^2)+10ab. \end{aligned} \tag{6.14}$$

In view of (6.2) ,observe that

$$w=4(a^2-6b^2) \tag{6.15}$$

Thus, (6.12),(6.14) and (6.15) satisfy (6.1).

Remark 6.2

It is to be noted that the integer 1 in (6.11) may be considered as

$$1=\frac{(6r^2+s^2+2rs\sqrt{6})(6r^2+s^2-2rs\sqrt{6})}{(6r^2-s^2)^2}$$

Repeating the above process and taking different values to r& s, one obtains different sets of integer solutions to (6.1).

Procedure 3

Introduction of the transformations

$$x = 27\beta, y = 3\gamma + 15\beta, z = 3\gamma + 6\beta, w = 9\delta \qquad (6.16)$$

in (6.1) leads to the Pythagorean equation

$$\gamma^2 = \delta^2 + \beta^2$$

Employing the most cited solutions of the above Pythagorean equation in (6.16),

the corresponding integer solutions to (6.1) are obtained.

Note 6.3

In (6.16) ,if we choose $x = 9\beta$,then (6.1) reduces to the Pythagorean equation

$$\gamma^2 = \delta^2 + 9\beta^2$$

which is satisfied by

$$\beta = 2rs, \delta = 9r^2 - s^2, \gamma = 9r^2 + s^2 \ ,3r > s > 0$$

In this case , the corresponding integer solutions to (6.1) are given by

$$x = 18rs, y = 3(9r^2 + s^2) + 30rs, z = 3(9r^2 + s^2) + 12rs, w = 9(9r^2 - s^2)$$

Procedure 4

The option

$$w = x + y \qquad (6.17)$$

in (6.1) leads to the ternary homogeneous quadratic equation

$$7y^2 + 2xy - 15z^2 = 0 \qquad (6.18)$$

Treating (6.18) as a quadratic equation in y and solving for the same ,we have

$$y = \frac{-x \pm \sqrt{105z^2 + x^2}}{7} \qquad (6.19)$$

The square-root on the R.H.S. of (6.19) is removed when

$$z = 2pq, x = 105p^2 - q^2 \quad (6.20)$$

Taking the negative sign before the square-root in (6.19) and from (6.17) ,we get

$$y = -30p^2, w = 75p^2 - q^2 \quad (6.21)$$

Thus,(6.20) & (6.21) satisfy (6.1).

Also, considering positive sign before the square-root in (6.19) & taking (6.17) ,

we, after some algebra, obtain the integer solutions to (6.1) to be

$$z = 14p\alpha, x = 105p^2 - 49\alpha^2, y = 14\alpha^2, w = 105p^2 - 35\alpha^2$$

Note 6.4

In addition to the above patterns of integer solutions ,there are some more choices of solutions to (6.1) which we illustrate as follows:

Let

$$\alpha^2 = x^2 + 105z^2 \quad (6.22)$$

Represent (6.22) as the system of double equations as shown in Table 6.1:

Table 6.1-System of double equations

System	I	II	III	IV	V	VI	VII	VIII
$\alpha + x$	$105z$	$35z$	$21z$	$15z$	$105z^2$	$35z^2$	$21z^2$	$15z^2$
$\alpha - x$	z	$3z$	$5z$	$7z$	1	3	5	7

Solving each of the above system of double equations, the values of α, x, z are obtained. From (6.19) and (6.17) ,the corresponding values to y, w satisfying (6.1) are found. For simplicity and brevity, the integer solutions to (6.1) obtained from each of the above system of double equations are exhibited.

Solutions to 6.1) from System I:

Set 1 $\quad x = 364s, y = s, z = 7s, w = 365s$

Set 2 $\quad x = 52s, y = -15s, z = s, w = 37s$

Solutions to (6.1) from System II:

Set 3 $x = 112s, y = 3s, z = 7s, w = 115s$

Set 4 $x = 16s, y = -5s, z = s, w = 11s$

Solutions to (6.1) from System III:

Set 5 $x = 56s, y = 5s, z = 7s, w = 61s$

Set 6 $x = 8s, y = -3s, z = s, w = 9s$

Solutions to (6.1) from System IV:

Set 7 $x = 28s, y = -15s, z = 7s, w = 13s$

Solutions to (6.1) from System V:

Set 8

$$x = 210s^2 + 210s + 52, y = -60s^2 - 60s - 15, z = 2s + 1, w = 150s^2 + 150s + 37$$

Solutions to (6.1) from System VI:

Set 9 $x = 70s^2 + 70s + 16, y = -20s^2 - 20s - 5, z = 2s + 1, w = 50s^2 + 50s + 11$

Solutions to (6.1) from System VII:

Set 10 $x = 42s^2 + 42s + 8, y = -12s^2 - 12s - 3, z = 2s + 1, w = 30s^2 + 30s + 5$

Solutions to (6.1) from System VIII:

Set 11 $x = 30s^2 + 30s + 4, y = 1, z = 2s + 1, w = 30s^2 + 30s + 5$

Set 12 $$\begin{aligned} &x = 30(7s-4)(7s-3) + 4, y = -105(2s-1)^2, z = 14s - 7, \\ &w = [30(7s-4)(7s-3) + 4] - 105(2s-1)^2 \end{aligned}$$

Procedure 5

The option

$$w = x - 4z \qquad (6.23)$$

in (6.1) leads to the ternary homogeneous quadratic equation

$$6y^2 - 8xz + z^2 = 0 \tag{6.24}$$

Treating (6.24) as a quadratic equation in z and solving for the same ,we have

$$z = \frac{8x \pm \sqrt{64x^2 - 24y^2}}{2} \tag{6.25}$$

The square-root on the R.H.S. of (6.25) is removed when

$$y = 8pq, x = 6p^2 + q^2 \tag{6.26}$$

Taking the negative sign before the square-root in (6.25) and from (6.23) ,we get

$$z = 8q^2, w = 6p^2 - 31q^2 \tag{6.27}$$

Thus,(6.26) & (6.27) satisfy (6.1).

Also, considering positive sign before the square-root in (6.25) & taking (6.23) , we, after some algebra, obtain

$$z = 48p^2, w = -186p^2 + q^2 \tag{6.28}$$

Thus,(6.26) & (6.28) satisfy (6.1).

Procedure 6

The substitution

$$x = 2k + 1, w = 2k - 1 \tag{6.29}$$

in (6.1) gives

$$8k = 6y^2 - 15z^2$$

The choice

$$y = 4Y, z = 4P \tag{6.30}$$

in the above equation gives

$$k = 12Y^2 - 30P^2$$

In view of (6.29) ,we get

$$\begin{aligned} x &= 24Y^2 - 60P^2 + 1, \\ w &= 24Y^2 - 60P^2 - 1. \end{aligned} \qquad (6.31)$$

Thus,(6.30) & (6.31) satisfy (6.1).

Procedure 7

The introduction of the linear transformation

$$w = 3y \qquad (6.32)$$

in (6.1) leads to the homogeneous ternary quadratic equation

$$x^2 + 15z^2 = 15y^2 \qquad (6.33)$$

Assume

$$y = a^2 + 15b^2 \qquad (6.34)$$

Express the integer 15 on the R.H.S. of (6.33) as

$$15 = (i\sqrt{15})(-i\sqrt{15}) \qquad (6.35)$$

Substituting (6.34) &(6.35) in (6.33) and applying factorization, consider

$$x + i\sqrt{15}z = (i\sqrt{15})(a + i\sqrt{15}b)^2$$

On comparing the coefficients of corresponding terms ,we get

$$x = -30ab, z = a^2 - 15b^2 \qquad (6.36)$$

From (6.32) ,one has

$$w = 3(a^2 + 15b^2) \qquad (6.37)$$

Thus, (6.34),(6.36) & (6.37) satisfy (6.1).

Note 6.5

Observe that (6.33) is also written in the form of ratio as

$$\frac{x}{5(y+z)} = \frac{3(y-z)}{x} = \frac{P}{Q}, Q \neq 0$$

Solving the above system of double equations ,we have

$$x = 30PQ, y = 3Q^2 + 5P^2, z = 3Q^2 - 5P^2 \tag{6.38}$$

From (6.32), we get

$$w = 3(3Q^2 + 5P^2) \tag{6.39}$$

Thus, (6.38) & (6.39) satisfy (6.1).

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Chapter 7

A scrutiny of integer solutions to homogeneous quaternary quadratic equation

7.1 Method of Analysis:

The homogeneous quadratic equation with four unknowns to be solved for its integer solutions is

$$2xy + 3z^2 = 8w^2 \tag{7.1}$$

We present below different sets of distinct integer solutions to (7.1) through employing linear transformations.

Introduction of the linear transformations

$$x = u + v, y = u - v, z = v, (u \neq v \neq 0) \tag{7.2}$$

in (7.1) leads to

$$v^2 + 2u^2 = 8w^2 \tag{7.3}$$

Assume

$$w = a^2 + 2b^2 \tag{7.4}$$

Set I

Write 8 as

$$8 = (i2\sqrt{2})(-i2\sqrt{2}) \tag{7.5}$$

Using (7.4) and (7.5) in (7.3) and employing the method of factorization,

define

$$v+i\sqrt{2}u=(i2\sqrt{2})(a+i\sqrt{2}b)^2$$

On equating the real and imaginary parts,one obtains

$$v=8ab, u=2a^2-4b^2$$

In view of (7.2),note that

$$\left.\begin{aligned} x&=2a^2-4b^2+8ab \\ y&=2a^2-4b^2-8ab \\ z&=8ab \end{aligned}\right\} \tag{7.6}$$

Thus, (7.6) and (7.4) represent the distinct integer solutions to (7.1).

Set II

Note that 8 may be expressed as the product of complex conjugates as below:

$$8=\frac{(8+i2\sqrt{2})(8-i2\sqrt{2})}{9} \tag{7.7}$$

Following the procedure as in Set I,the corresponding integer solutions to (7.1) are given below:

$$\begin{aligned} x&=3(10a^2-20b^2+8ab) \\ y&=3(-6a^2+12b^2+24ab) \\ z&=3(8a^2-16b^2-8ab) \\ w&=9(a^2+2b^2) \end{aligned}$$

Set III

(7.3) is written as

$$v^2 + 2u^2 = 8w^2 = 8w^2 * 1 \tag{7.8}$$

Consider 1 as

$$1 = \frac{(1 + i2\sqrt{2})(1 - i2\sqrt{2})}{9} \tag{7.9}$$

Using (7.9), (7.5) and (7.4) in (7.8) and employing the method of factorization, define

$$v + i\sqrt{2}u = (i2\sqrt{2})(a + i\sqrt{2}b)^2 \frac{(1 + i2\sqrt{2})}{3}$$

In this case,the corresponding integer solutions to (7.1) are found to be

$$x = 3(-6a^2 + 12b^2 - 24ab)$$
$$y = 3(10a^2 - 20b^2 - 8ab)$$
$$z = 3(-8a^2 + 16b^2 - 8ab)$$
$$w = 9(a^2 + 2b^2)$$

It is worth to note that, by substituting (7.9),(7.7) and (7.4) in (7.8) and performing the analysis as above,one obtains a different set of integer solutions to (7.1).

Remark

It is worthmentioning here that, in (7.9),1 may be represented as the product of complex conjugates,in general, as exhibited below:

$$1 = \frac{(2r^2 - s^2 + i2\sqrt{2}rs)(2r^2 - s^2 - i2\sqrt{2}rs)}{(2r^2 + s^2)^2}$$

Set IV

Introduction of the linear transformations

$$x = X + 8T + 6V, y = X + 8T - 6V, z = 6V, w = X + 2T \tag{7.10}$$

in (7.1) leads to

$$X^2 = 16T^2 + 6V^2 \tag{7.11}$$

After performing a few calculations,the above equation is satisfied by

the following threechoices of solutions:

i. $X = 20k, T = k, V = 8k$

ii. $X = 28k, T = 5k, V = 8k$

iii. $X = 24R^2 + 4S^2, T = 6R^2 - S^2, V = 8RS$

In view of (7.10),the corresponding integer solutions to (7.1) are represented

as follows:

Solutions for (i):

$$x = 76k, y = -20k, z = 48k, w = 22k$$

Solutions for (ii):

$$x = 116k, y = 20k, z = 48k, w = 38k$$

Solutions for (iii):

$$x = 72R^2 - 4S^2 + 48RS, y = 72R^2 - 4S^2, V - 48RS, z = 48RS, w = 36R^2 + 2S^2$$

Note: Suppose,instead of (7.10),the linear transformations are taken as

$$x = X - 8T + 6V, y = X - 8T - 6V, z = 6V, w = X - 2T$$

then,the corresponding three choices of solutions to (7.1) are as follows:

Solutions for (i):

$$x = 60k, y = -36k, z = 48k, w = 18k$$

Solutions for (ii):

$$x = 36k, y = -60k, z = 48k, w = 18k$$

Solutions for (iii):

$$x = -24R^2 + 12S^2 + 48RS, y = -24R^2 + 12S^2 - 48RS, z = 48RS, w = 12R^2 + 6S^2$$

Generation of solutions

Three different formulas for generating sequence of integer solutions based on the given solution are presented below:

Let $\left(x_0, y_{0,} z_0, w_0\right)$ be any given solution to (7.1)

Formula: 1

Let $\left(x_{1,} y_1, z_1, w_1\right)$ given by

$$x_1 = x_0,\ y_1 = y_0, z_1 = 2h - z_0, w_1 = h + w_0 \qquad (7.12)$$

be the 2[nd]solution to (7.1). Using (7.12) in (7.1) and simplifying, one obtains

$$h = 3z_0 + 4w_0$$

In view of (7.12), the values of z_1 and w_1 are written in the matrix form as

$$\left(z_1, w_1\right)^t = M\left(z_0, w_0\right)^t \qquad (7.13)$$

where

$$M = \begin{pmatrix} 5 & 8 \\ 3 & 5 \end{pmatrix} \text{ and } t \text{ is the transpose}$$

The repetition of the above process leads to the n[th] solutions z_n, w_n given by

$$(z_n, w_n)^t = M^n (z_0, w_0)^t$$

We know that

$$M^n = \frac{a^n}{(\alpha-\beta)}(M-\beta I) + \frac{\beta^n}{(\beta-\alpha)}(M-\alpha I),$$

$I = 2\times 2$ Identity matrix and α, β are the distinct Eigen values of *M*.

For *M* given above in (7.13), It is seen that

$$\alpha = 5+2\sqrt{6},\ \beta = 5-2\sqrt{6}$$

Thus, the generation formula to obtain sequence of integer solutions to (7.1) is given by

$$\begin{aligned} x_n &= x_0, y_n = y_0 \\ z_n &= \left(\frac{\alpha^n+\beta^n}{2}\right) z_0 + \frac{2}{\sqrt{6}}(\alpha^n-\beta^n) w_0 \\ w_n &= \frac{3}{4\sqrt{6}}(\alpha^n-\beta^n) z_0 + \left(\frac{\alpha^n+\beta^n}{2}\right) w_0 \end{aligned}$$

Formula: 2

Let (x_1, y_1, z_1, w_1) given by

$$x_1 = 5x_0,\ y_1 = 5y_0,\ z_1 = 5z_0 + h,\ w_1 = h - 5w_0 \tag{7.14}$$

be the 2[nd] solution to (7.1). For this choice, the generation formula for getting sequence of integer solutions to (7.1) is obtained as below:

$$x_n = 5^n x_0, y_n = 5^n y_0$$

where $$z_n = \left(\frac{\alpha^n + \beta^n}{2}\right) z_0 + \frac{2}{\sqrt{6}}(\alpha^n - \beta^n) w_0$$

$$w_n = \frac{\sqrt{6}}{8}(\alpha^n - \beta^n) z_0 + \left(\frac{\alpha^n + \beta^n}{2}\right) w_0$$

$$\alpha = 11 + 4\sqrt{6},\ \beta = 11 - 4\sqrt{6}$$

Formula: 3

Let (x_1, y_1, z_1, w_1) given by

$$x_1 = 3h - x_0,\ y_1 = h - y_0,\ z_1 = -z_0 + h,\ w_1 = h + w_0 \tag{7.15}$$

be the 2nd solution to (7.1).Using (7.15) in (7.1) and simplifying, one obtains

$$h = 2x_0 + 6y_0 + 6z_0 + 16w_0$$

In view of (7.15), we have

$$x_1 = 5x_0 + 18y_0 + 18z_0 + 48w_0$$
$$y_1 = 2x_0 + 5y_0 + 6z_0 + 16w_0$$
$$z_1 = 2x_0 + 6y_0 + 5z_0 + 16w_0$$
$$w_1 = 2x_0 + 6y_0 + 6z_0 + 17w_0$$

which is written in the form of matrix as

$$(x_1, y_{1,} z_1, w_1)^t = \begin{pmatrix} 5 & 18 & 18 & 48 \\ 2 & 5 & 6 & 16 \\ 2 & 6 & 5 & 16 \\ 2 & 6 & 6 & 17 \end{pmatrix} (x_0, y_0, z_0, w_0)^t$$

where t is the transpose. The repetition of the above process leads to the general solution to (7.1) as

$$x_{n+1} = \frac{Y_n - (-1)^n 2}{3} x_0 + (Y_n + (-1)^n) y_0 + (Y_n + (-1)^n) z_0 + 8X_n w_0$$

$$y_{n+1} = \frac{Y_n + (-1)^n}{9} x_0 + \frac{Y_n - (-1)^n 2}{3} y_0 + \frac{Y_n + (-1)^n}{3} z_0 + \frac{8X_n}{3} w_0$$

$$z_{n+1} = \frac{Y_n + (-1)^n}{9} x_0 + \frac{Y_n + (-1)^n}{3} y_0 + \frac{Y_n - (-1)^n 2}{3} z_0 + \frac{8X_n}{3} w_0$$

$$w_{n+1} = \frac{X_n}{3} x_0 + X_n y_0 + X_n z_0 + Y_n w_0$$

where

$$Y_n = \frac{1}{2}\left(\left(17 + 6\sqrt{8}\right)^{n+1} + \left(17 - 6\sqrt{8}\right)^{n+1}\right),$$

$$X_n = \frac{1}{2\sqrt{8}}\left(\left(17 + 6\sqrt{8}\right)^{n+1} - \left(17 - 6\sqrt{8}\right)^{n+1}\right)), n = 0,1,2,\ldots$$

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Chapter 8

A portrayal of integer solutions to non-homogeneous quaternary quadratic equation

8.1 Method of Analysis:

The non-homogeneous quadratic equation with four unknowns to be solved is

$$.\, x^2 + y^2 + z^2 = t^2 - 1 \tag{8.1}$$

Different patterns of solutions in integers are illustrated below:

Pattern 1

The choice

$$t = x + 1 \tag{8.2}$$

in (8.1) gives

$$x = \frac{(y^2 + z^2)}{2} \tag{8.3}$$

As our aim is to find integer solutions , observe that the values of x are integers when y,z are both odd or even and from (8.2) , the corresponding values of t are obtained. The two sets of integer solutions are presented explicitly below:

Set 1

$$y = 2r, z = 2s, x = 2(r^2 + s^2), t = 2(r^2 + s^2) + 1$$

Set 2

$$y = 2r + 1, z = 2s + 1, x = 2(r^2 + s^2 + r + s) + 1, t = 2(r^2 + s^2 + r + s + 1)$$

Pattern 2

The substitution

$$t = x + y, z = 2k + 1 \tag{8.4}$$

in (8.1) leads to

$$x\,y = 2k^2 + 2k + 1 \tag{8.5}$$

For any given k in (8.5) , one may determine the values of x and y. In view of (8.4) , the corresponding values of z ,t are obtained. For the sake of clear understanding, a few numerical solutions are exhibited in Table 8.1 below:

Table 8.1-Numerical solutions

k	z	x*y	X	y	t
1	3	5	5	1	6
5	11	61	1	61	62
8	17	145	145	1	146
8	17	145	5	29	34
-3	-5	13	1	13	14
-10	-19	81	81	1	82
-10	-19	81	27	3	30

Pattern 3

Introducing the transformations

$$x = u + v,\, y = u - v,\, z = 2v,\, t = 2u + 1 \tag{8.6}$$

in (8.1), it simplifies to

$$(u+1)^2 = 3v^2 + 1 \tag{8.7}$$

which is satisfied by

$$u_n = \frac{f_n}{2} - 1,\, v_n = \frac{g_n}{2\sqrt{3}} \tag{8.8}$$

where

$$f_n = (2+\sqrt{3})^{n+1} + (2-\sqrt{3})^{n+1},$$
$$g_n = (2+\sqrt{3})^{n+1} - (2-\sqrt{3})^{n+1}.$$

In view of (8.6), the non-zero distinct integer values of the quadruple (x_n, y_n, z_n, t_n) aregiven by

$$x_n = \frac{(\sqrt{3}\,f_n + g_n)}{2\sqrt{3}} - 1,$$

$$y_n = \frac{(\sqrt{3}\,f_n - g_n)}{2\sqrt{3}} - 1, \qquad (8.9)$$

$$z_n = \frac{g_n}{\sqrt{3}}, t_n = f_n - 1 .$$

The above values of x_n, y_n, z_n and t_n given by (8.9) satisfy the following recurrence relations

$$x_{n+2} - 4x_{n+1} + x_n = 2 ,$$
$$y_{n+2} - 4y_{n+1} + y_n = 2 ,$$
$$z_{n+2} - 4z_{n+1} + z_n = 0,$$
$$t_{n+2} - 4t_{n+1} + t_n = 2 .$$

Generation of solutions

An interesting question that one often raises is: Given an initial solution, whether one can obtain a general formula for generating a sequence of solutions? The answer to this question is YES and the procedure is as follows:

Let (x_0, y_0, z_0, t_0) be the given non-zero distinct initial integer quadruple satisfying (8.1).

Let h be any non-zero integer such that

$$x_1 = h - x_0, y_1 = h - y_0, z_1 = h - z_0, t_1 = h + t_0 \qquad (8.10)$$

be the second solution of (8.1). Substituting (8.10) in (8.1), the value of h is found to be

$$h = x_0 + y_0 + z_0 + t_0 \qquad (8.11)$$

In view of (8.10) the second solution generated from the initial solution is written in thematrix form as

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \\ t_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \\ z_0 \\ t_0 \end{pmatrix} = \begin{pmatrix} a(1) & b(1) & b(1) & c(1) \\ b(1) & a(1) & b(1) & c(1) \\ b(1) & b(1) & a(1) & c(1) \\ c(1) & c(1) & c(1) & d(1) \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \\ z_0 \\ t_0 \end{pmatrix} \qquad (8.12)$$

where $a(1) = 0, b(1) = 1, c(1) = 1, d(1) = 2$

The repetition of the above process leads to the general form for the generated solution which is written in the matrix form as follows:

$$\begin{pmatrix} x_n \\ y_n \\ z_n \\ t_n \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 2 \end{pmatrix}^n \begin{pmatrix} x_0 \\ y_0 \\ z_0 \\ t_0 \end{pmatrix} = \begin{pmatrix} a(n) & b(n) & b(n) & c(n) \\ b(n) & a(n) & b(n) & c(n) \\ b(n) & b(n) & a(n) & c(n) \\ c(n) & c(n) & c(n) & d(n) \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \\ z_0 \\ t_0 \end{pmatrix} \qquad (8.13)$$

where

$$a(n) = 2b(n-1) + c(n-1)$$

$$b(n) = b(n-1) + c(n-1) + a(n-1)$$

$$c(n) = 2c(n-1) + d(n-1)$$

$$d(n) = 3(2b(n-1) + c(n-1)) + 2(-1)^n, n > 1.$$

The above system of equation (8.13) is computed numerically using **C++ program** and a few solutions generated are presented in Table 8.2.

Table 8.2 -SOLUTIONS OF $X^2 + Y^2 + Z^2 = T^2 - 1$

I	a(i)	b(i)	c(i)	d(i)	x(i)	y(i)	z(i)	t(i)
0	-	-	-	-	5	1	3	6
1	0	1	1	2	10	14	12	21
2	3	2	4	7	47	43	45	78
3	8	9	15	26	166	170	168	291
4	33	32	56	97	629	625	627	1086
5	120	121	209	362	2338	2342	2340	4053
6	451	450	780	1351	8735	8731	8733	15126
7	1680	1681	2911	5042	32590	32594	32592	56451
8	6273	6272	10864	18817	121637	121633	121635	210678
9	3408	23409	40545	70226	453946	453950	453948	786261
10	87363	87362	151316	262087	1694159	1694155	1694157	2934366
11	326040	326041	564719	978122	6322678	6322682	6322680	109512

								03
12	1216801	1216800	2107560	3650401	23596565	23596561	23596563	40870446
13	4541160	4541161	865521	13623482	88063570	88063574	88063572	152530581
14	16947843	16947842	29354524	50843527	328657727	328657723	328657725	56925187

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Chapter 9

Designs of integer solutions to non-homogeneous quaternary quadratic equation

9.1 Method of Analysis:

The non-homogeneous quadratic equation with four unknowns to be solved is

$$.\, x^2 + y^2 + z^2 = t^2 + 1 \tag{9.1}$$

Different patterns of solutions in integers are illustrated below:

Pattern 1

The choice

$$z = t - k, k > 0 \tag{9.2}$$

in (9.1) gives

$$t = \frac{(x^2 + y^2 + k^2 - 1)}{2k} \tag{9.3}$$

As our aim is to find integer solutions, observe that the values of t are integers when x,y are chosen suitably and from (9.2) , the corresponding values of z are obtained . For the sake of clear understanding, four sets of integer solutions are exhibited below:

Set 1

$$x = (2r+1)k, y = 2sk+1, t = 2(r^2+s^2+r)k+2s+k\ , z = 2(r^2+s^2+r)k+2s$$

Set 2

$$x = 2rk, y = (2s-1)k+1, t = 2(r^2+s^2-s)k+k+2s-1, z = 2(r^2+s^2-s)k+2s-1$$

Set 3

$$x = Nk, y = (N-1)k+1, t = (N^2-N+1)k+N-1, z = (N^2-N)k+N-1$$

Set 4

$$x=(2s-1)k,\ y=2sk+1,\ t=(4s^2-2s+1)k+2s,\ z=(4s^2-2s)k+2s$$

Pattern 2

The substitution

$$t=x+y,\ z=2k+1 \tag{9.4}$$

in (9.1) leads to

$$xy=2k^2+2k,\ k\neq -1 \tag{9.5}$$

For any given k in (9.5) , one may determine the values of x and y. In view of (9.4) , the corresponding values of z ,t are obtained. For the sake of clear understanding, a few numerical solutions are exhibited in Table 9.1 below:

Table 9.1-Numerical solutions

k	z	x*y	x	y	t
k	2 k+1	2 k (k+1)	2	k (k+1)	k^2+k+2
			k	2 (k+1)	3 k+2
			2 k	k+1	3 k+1
			k (k+1)	2	K^2+k+2
			2 k (k+1)	1	2 k^2+2 k+1

Pattern 3

Introducing the transformations

$$x=u+v,\ y=u-v,\ z=2u+1,\ t=2v \tag{9.6}$$

in (9.1), it simplifies to

$$(3u+1)^2=3v^2+1 \tag{9.7}$$

which is satisfied by

$$u_n=\frac{(f_n-2)}{6},\ v_n=\frac{g_n}{2\sqrt{3}} \tag{9.8}$$

where

$$f_n = (2+\sqrt{3})^{n+1} + (2-\sqrt{3})^{n+1},$$
$$g_n = (2+\sqrt{3})^{n+1} - (2-\sqrt{3})^{n+1}.$$

In view of (9.6), the non-zero distinct integer values of the quadruple (x_n, y_n, z_n, t_n) aregiven by

$$x_n = \frac{(f_n + \sqrt{3}g_n - 2)}{6},$$
$$y_n = \frac{(f_n - \sqrt{3}g_n - 2)}{6}, \tag{9.9}$$
$$z_n = \frac{(f_n + 1)}{3}, t_n = \frac{g_n}{\sqrt{3}}, n = -1,1,3,...$$

A few numerical solutions to (9.1) are given below:

$x_1 = 6, y_1 = -2, z_1 = 5, t_1 = 8$

$x_3 = 88, y_3 = -24, z_3 = 65, t_3 = 112$

$x_5 = 1230, y_5 = -330, z_5 = 901, t_5 = 1560$

The above values of x_n, y_n, z_n and t_n given by (9.9) satisfy the following recurrence relations

$x_{n+4} - 14x_{n+2} + x_n = 4$,

$y_{n+4} - 14y_{n+2} + y_n = 4$,

$z_{n+4} - 14z_{n+2} + z_n = -4$,

$t_{n+4} - 14t_{n+2} + t_n = 0$.

Pattern 4

Taking

$$x = 2rs, y = r^2 - s^2\ (r > s > 0), t = u + v, z = u - v, (u \neq v) \tag{9.10}$$

in (9.1) ,we have

$$(r^2 + s^2)^2 = 4uv + 1 \tag{9.11}$$

It is worth to note that the values of r,s should be of different parity as the R.H.S. of (9.11) is odd. The choice

$$r = 2p+1, s = 2q, p > q > 0 \tag{9.12}$$

in (9.11) gives

$$uv = (4p^2 + 4p + 4q^2 + 2)(p^2 + p + q^2)$$

Choose

$$u = (4p^2 + 4p + 4q^2 + 2),\ v = (p^2 + p + q^2) \tag{9.13}$$

Employing (9.12) & (9.13) in (9.10) ,the corresponding integer solutions to (9.1) are given by

$$x = 4q(2p+1),$$
$$y = 4p^2 + 4p + 1 - 4q^2,$$
$$z = 3p^2 + 3p + 2 + 3q^2,$$
$$t = 5p^2 + 5p + 2 + 5q^2.$$

Pattern 5

The substitution

$$x = u + v, y = u - v, t = 2u, z^2 = 2w^2 + 1 \tag{9.14}$$

in (9.1) leads to the Pythagorean equation

$$v^2 + w^2 = u^2 \tag{9.15}$$

which is satisfied by

$$u = p^2 + q^2, v = p^2 - q^2, w = 2pq, p > q > 0 \tag{9.16}$$

Substituting the value of w from (9.16) in (9.14) ,we have

$$z^2 = 8X^2 + 1 \tag{9.17}$$

where

$$X = pq \tag{9.18}$$

The general solution (X_n, z_n) to the pellian equation (9.17) is given by

$$X_n = \frac{1}{2\sqrt{8}} g_n \, , z_n = \frac{1}{2} f_n \qquad (9.19)$$

where
$$f_n = (3+\sqrt{8})^{n+1} + (3-\sqrt{8})^{n+1} ,$$
$$g_n = (3+\sqrt{8})^{n+1} - (3-\sqrt{8})^{n+1} .$$

In view of (9.18) , we have from (9.19)

$$\begin{aligned} X_n = p_n * q_n &= \frac{1}{2\sqrt{8}} [(3+\sqrt{8})^{n+1} - (3-\sqrt{8})^{n+1}] \\ &= \frac{1}{4\sqrt{2}} [(\sqrt{2}+1)^{2n+2} - (\sqrt{2}-1)^{2n+2}] \\ &= \frac{1}{2} [(\sqrt{2}+1)^{n+1} + (\sqrt{2}-1)^{n+1}] * \frac{1}{2\sqrt{2}} [(\sqrt{2}+1)^{n+1} - (\sqrt{2}-1)^{n+1}] \end{aligned}$$

Thus ,

$$p_n = \frac{1}{2} [(\sqrt{2}+1)^{n+1} + (\sqrt{2}-1)^{n+1}] \, ,$$
$$q_n = \frac{1}{2\sqrt{2}} [(\sqrt{2}+1)^{n+1} - (\sqrt{2}-1)^{n+1}].$$

From (9.14) ,the corresponding integer solutions to (9.1) are given by

$$x_n = \frac{1}{2} [(\sqrt{2}+1)^{n+1} + (\sqrt{2}-1)^{n+1}]^2 \, ,$$
$$y_n = \frac{1}{4} [(\sqrt{2}+1)^{n+1} - (\sqrt{2}-1)^{n+1}]^2 \, ,$$
$$z_n = \frac{1}{2} [(\sqrt{2}+1)^{2n+2} + (\sqrt{2}-1)^{2n+2}],$$
$$t_n = \frac{1}{4} [3\,(\sqrt{2}+1)^{2n+2} + 3\,(\sqrt{2}-1)^{2n+2} + 2] \, , n = 0,1,2,3,... \qquad (9.20)$$

The above values of x_n , y_n , z_n and t_n given by (9.20) satisfy the following recurrence relations

$$x_{n+2} - 6x_{n+1} + x_n = -4\,,$$
$$y_{n+2} - 6y_{n+1} + y_n = 2\,,$$
$$z_{n+2} - 6z_{n+1} + z_n = 0,$$
$$t_{n+2} - 6t_{n+1} + t_n = -2\,.$$

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Chapter 10

Patterns on integer solutions to homogeneous quaternary quadratic equation

The quadratic equation with four unknowns to be solved is

$$x^2 + y^2 = 2\left(z^2 - w^2\right) \tag{10.1}$$

There are two types of integer solutions .In Section A , real integer solutions are obtained. In Section B , the Gaussian integer solutions are determined.

Section A ; Real Integer Solutions

Introduction of the linear transformations

$$x = u + v, \quad y = u - v \tag{10.2}$$

in (10.1) leads to

$$u^2 + v^2 + w^2 = z^2 \tag{10.3}$$

which is in the form of space Pythagorean equation

The choices of solutions for (10.3) are represented below:

i) $u = m^2 - n^2 - p^2 + q^2$, $v = 2mn - 2pq$, $w = 2mp + 2nq$, $z = m^2 + n^2 + p^2 + q^2$

ii) $u = 2mp + 2nq$, $v = 2mn - 2pq$, $w = m^2 - n^2 - p^2 + q^2$, $z = m^2 + n^2 + p^2 + q^2$

iii) $u = 2mp + 2nq$, $v = m^2 - n^2 - p^2 + q^2$, $w = 2mn - 2pq$, $z = m^2 + n^2 + p^2 + q^2$

iv) $u = 2ab$, $v = 2ac$, $w = a^2 - b^2 - c^2$, $z = a^2 + b^2 + c^2$

v) $u = a^2 - b^2 - c^2$, $v = 2ac$, $w = 2ab$, $z = a^2 + b^2 + c^2$

In view of (10.2), one may obtain different sets of solutions to (10.1) which are presented below:

Set: 1

Considering choice (i), the general solution of (10.1) is

$$x = m^2 - n^2 - p^2 + q^2 + 2mn - 2pq$$

$$y = m^2 - n^2 - p^2 + q^2 - 2mn + 2pq$$

$$z = m^2 + n^2 + p^2 + q^2$$

$$w = 2mp + 2nq$$

Set: 2

For choice (ii), the general solution of (10.1) is

$$x = 2m(p+n) + 2q(n-p)$$

$$y = 2m(p-n) + 2q(n+p)$$

$$z = m^2 + n^2 + p^2 + q^2$$

$$w = m^2 - n^2 - p^2 + q^2$$

Set: 3

For choice (iii), the general solution of (10.1) is

$$x = m^2 - n^2 - p^2 + q^2 + 2mp + 2nq$$

$$y = -m^2 + n^2 + p^2 - q^2 + 2mp + 2nq$$

$$z = m^2 + n^2 + p^2 + q^2$$

$$w = 2mn - 2pq$$

Set: 4

For choice (iv), the general solution of (10.1) is

$$x = 2a(b+c)$$

$$y = 2a(b-c)$$

$$z = a^2 + b^2 + c^2$$

$$w = a^2 - b^2 - c^2$$

Set: 5

For choice (v), the general solution of (10.1) is

$$x = a^2 - b^2 - c^2 + 2ac$$

$$y = a^2 - b^2 - c^2 - 2ac$$

$$z = a^2 + b^2 + c^2$$

$$w = 2ab$$

In addition to the above sets of solutions to (10.1), there are other representations of solutions to (10.1) which are illustrated below:

Representation: 1

Write (10.3) as

$$u^2 + v^2 = z^2 - w^2 \tag{10.4}$$

Assume $\alpha^2 = z^2 - w^2$ (10.5)

Rewrite (10.5) as

$$z^2 = \alpha^2 + w^2 \tag{10.6}$$

which is in the form of Pythagorean equation satisfied by the following two sets of solutions

Set: 1 $z = r^2 + s^2$, $w = 2rs$, $\alpha = r^2 - s^2$, $r > s > 0$ (10.7)

Set: 2 $z = r^2 + s^2$, $w = r^2 - s^2$, $\alpha = 2rs$, $r > s > 0$ (10.8)

Consider (10.7). Using (10.7) in (10.4), we have

$$u^2 + v^2 = \left(r^2 - s^2\right)^2 \tag{10.9}$$

which is satisfied by

$$r = f^2 + g^2 + h^2$$

$$s = f^2 - g^2 - h^2$$

$$u = 8f^2gh$$

$$v = 4f^2\left(g^2 - h^2\right)$$

In view of (10.2) and (10.7), the corresponding non-zero distinct integral solutions of (10.1) are given by

$$x = 4f^2\left(2gh + g^2 - h^2\right)$$

$$y = 4f^2\left(2gh - g^2 + h^2\right)$$

$$z = 2\left(f^4 + g^4 + h^4 + 2g^2h^2\right)$$

$$w = 2\left(f^4 - g^4 - h^4 - 2g^2h^2\right)$$

Consider (10.8). Using (10.8) in (10.4), we have

$$u^2 + v^2 = \left(2rs\right)^2 \tag{10.10}$$

which is in the form of Pythagorean equation satisfied by

$$u = p^2 - q^2 \tag{10.11}$$

$$v = 2pq \tag{10.12}$$

$$2rs = p^2 + q^2 \tag{10.13}$$

Here, the equation (10.13) is satisfied for the following choices of r and s:

i) $s = 1$, $r = k^2 + 2k + 2$

ii) $s = 2$, $r = 2k^2 + 2k + 1$

iii) $s = 2$, $r = 2k^2 + 4k + 4$

Considering choice (i) and performing simplification, the corresponding solutions to (10.1) are given by

$$x = 2k^2 + 8k + 4$$

$$y = 4 - 2k^2$$

$$z = k^4 + 4k^3 + 8k^2 + 8k + 5$$

$$w = k^4 + 4k^3 + 8k^2 + 8k + 3$$

Similarly for choice (ii), the general solutions to (10.1) are given by

$$x = 8k^2 + 16k + 4$$

$$y = 4 - 8k^2$$

$$z = 4k^4 + 8k^3 + 8k^2 + 4k + 5$$

$$w = 4k^4 + 8k^3 + 8k^2 + 4k - 3$$

For choice (iii), the general solutions to (10.1) are found to be

$$x = 8k^2 + 32k + 16$$

$$y = 16 - 8k^2$$

$$z = 4k^4 + 16k^3 + 32k^2 + 32k + 20$$

$$w = 4k^4 + 16k^3 + 32k^2 + 32k + 12$$

Representation: 2

The assumption $\alpha^3 = z^2 - w^2$ (10.14)

is equivalent to the following system of double equations:

Table: 10.1 System of equations

System	$z+w$	$z-w$
1	α^2	α
2	α^3	1

Considering System: 1, it is seen that there are two sets of solutions to (10.1) represented respectively below:

Set: 1

$$x = (m+n)(m^2+n^2)$$

$$y = (m-n)(m^2+n^2)$$

$$z = \frac{1}{2}(m^2+n^2)(m^2+n^2+1)$$

$$w = \frac{1}{2}(m^2+n^2)(m^2+n^2-1)$$

Set: 2

$$x = m^2(m+3n) - n^2(3m+n)$$

$$y = m^2(m-3n) - n^2(3m-n)$$

$$z = \frac{1}{2}(m^2+n^2)(m^2+n^2+1)$$

$$w = \frac{1}{2}(m^2+n^2)(m^2+n^2-1)$$

Similarly, Considering System: 2, it is seen that there are two sets of solutions to (10.1) represented respectively below:

Set: 3

$$x = (m+n)(m^2+n^2)$$

$$y = (m-n)(m^2+n^2)$$

$$z = \frac{1}{2}\left[(m^2+n^2)^3+1\right]$$

$$w = \frac{1}{2}\left[(m^2+n^2)^3-1\right]$$

Set: 4

$$x = m^2(m+3n) - n^2(3m+n)$$

$$y = m^2(m-3n) - n^2(3m-n)$$

$$z = \frac{1}{2}\left[(m^2+n^2)^3+1\right]$$

$$w = \frac{1}{2}\left[(m^2+n^2)^3-1\right]$$

It is worth to note that m and n should be of different parity. Otherwise, the values of z and w are not in integers.

Representation: 3

Substituting $z=\frac{\alpha(\alpha+1)}{2}$ and $w=\frac{\alpha(\alpha-1)}{2}$ (10.15)

in (10.1), we get

$$x^2+y^2=2\alpha^3 \tag{10.16}$$

Assume $\alpha=p^2+q^2$, $p,q>0$ (10.17)

Write 2 as

$$2=(1+i)(1-i) \tag{10.18}$$

Substituting (10.17), (10.18) in (10.16) and employing the method of factorization, define

$$x+iy=(1+i)(p+iq)^3$$

Equating real and imaginary parts, we have

$$\left.\begin{aligned} x &= p^3-3p^2q-3pq^2+q^3 \\ y &= p^3+3p^2q-3pq^2-q^3 \end{aligned}\right\} \tag{10.19}$$

In view of (10.15), we have

$$\left.\begin{aligned} z &= \frac{1}{2}(p^2+q^2)(p^2+q^2+1) \\ w &= \frac{1}{2}(p^2+q^2)(p^2+q^2-1) \end{aligned}\right\} \tag{10.20}$$

Thus (10.19) and (10.20) represents non-zero distinct integral solutions to (10.1).

Note:

Instead of (10.18), one may write 2 as

$$2 = \frac{(7+i)(7-i)}{25}, \quad 2 = \frac{(1+7i)(1-7i)}{25}$$

Following the procedure similar to above, one may obtain different sets of integral solutions to (10.1).

Representation: 4

Substituting $z = \frac{\alpha^3+1}{2}$ and $w = \frac{\alpha^3-1}{2}$ (10.21)

in (10.1), we get (10.16).

Using (10.17) in (10.21)

$$\left.\begin{aligned} z &= \frac{1}{2}\left[\left(p^2+q^2\right)^3+1\right] \\ w &= \frac{1}{2}\left[\left(p^2+q^2\right)^3-1\right] \end{aligned}\right\} \qquad (10.22)$$

Hence (10.19) and (10.22) represents non-zero distinct integral solutions to (10.1). It is worth to note that p and q should be of different parity. Otherwise, the values of z and w are not in integers.

Section B: Gaussian integer solutions

The substitution

$$x = a + i2b, \; y = 2a - ic, \; z = b + ia, \; w = c + ia \qquad (10.23)$$

in (10.1) leads to

$$5a^2 + c^2 = 6b^2 \qquad (10.24)$$

(10.24) is solved through three different methods and thus we obtain three different sets of Gaussian integer solutions to (10.1)

Method: 1

Write (10.24) in the form of ratio as

$$\frac{5(a+b)}{b+c} = \frac{b-c}{a-b} = \frac{m}{n}, \quad n \neq 0 \tag{10.25}$$

which is equivalent to the system of double equations

$$5na + (5n-m)b - mc = 0$$
$$-ma + (m+n)b - nc = 0$$

Applying the method of cross multiplication, we get

$$a = m^2 - 5n^2 + 2mn \tag{10.26}$$

$$b = m^2 + 5n^2 \tag{10.27}$$

$$c = 5n^2 - m^2 + 10mn \tag{10.28}$$

In view of (10.23), the corresponding non-zero distinct Gaussian integer solutions of (10.1) are given by

$$x = m^2 - 5n^2 + 2mn + i(2m^2 + 10n^2)$$

$$y = 2m^2 - 10n^2 + 4mn - i(5n^2 - m^2 + 10mn)$$

$$z = m^2 + 5n^2 + i(m^2 - 5n^2 + 2mn)$$

$$w = 5n^2 - m^2 + 10mn + i(m^2 - 5n^2 + 2mn)$$

Method: 2

Assume $b = 5p^2 + q^2$, $p, q > 0$ (10.29)

Write 6 as

$$6 = \left(\sqrt{5} + i\right)\left(\sqrt{5} - i\right)$$

(10.30)

Substituting (10.29), (10.30) in (10.24) and employing the method of factorization, define

$$\sqrt{5}a + ic = \left(\sqrt{5} + i\right)\left(\sqrt{5}p + iq\right)^2 \tag{10.31}$$

Equating real and imaginary parts, we get

$$\left.\begin{aligned} a &= 5p^2 - q^2 - 2pq \\ c &= 5p^2 - q^2 + 10pq \end{aligned}\right\} \tag{10.32}$$

Using (10.29), (10.32) in (10.23), the corresponding non-zero distinct Gaussian integral solutions to (4.1) are found to be

$$x = 5p^2 - q^2 - 2pq + i\left(10p^2 + 2q^2\right)$$

$$y = 10p^2 - 2q^2 - 4pq - i\left(5p^2 - q^2 + 10pq\right)$$

$$z = 5p^2 + q^2 + i\left(5p^2 - q^2 - 2pq\right)$$

$$w = 5p^2 - q^2 + 10pq + i\left(5p^2 - q^2 - 2pq\right)$$

Method: 3

One may write (10.24) as

$$6b^2 - c^2 = 5a^2 \tag{10.33}$$

Assume $a = 6p^2 - q^2$, $p, q > 0$ (10.34)

Write 5 as

$$5 = \left(\sqrt{6} + 1\right)\left(\sqrt{6} - 1\right) \tag{10.35}$$

Substituting (10.34), (10.35) in (10.33) and employing the method of factorization, define

$$\sqrt{6}b + c = \left(\sqrt{6}+1\right)\left(\sqrt{6}p+q\right)^2 \tag{10.36}$$

Equating rational and irrational parts, we get

$$\left.\begin{aligned} b &= 6p^2 + q^2 + 2pq \\ c &= 6p^2 + q^2 + 12pq \end{aligned}\right\} \tag{10.37}$$

In view of (10.23), the corresponding non-zero distinct Gaussian integral solutions to (10.1) are given by

$$x = 6p^2 - q^2 + i\left(12p^2 + 2q^2 + 4pq\right)$$
$$y = 2\left(6p^2 - q^2\right) - i\left(6p^2 + q^2 + 12pq\right)$$
$$z = 6p^2 + q^2 + 2pq + i\left(6p^2 - q^2\right)$$
$$w = 6p^2 + q^2 + 12pq + i\left(6p^2 - q^2\right)$$

Generation of solutions:

Let (x_0, y_0, z_0) be the given integer solution to (10.1). Let (x_1, y_1, z_1) be the second solution of (10.1) where

$$x_1 = h - x_0 \text{ , } y_1 = h - y_0 \text{ , } z_1 = z_0 + h \text{ , } w_1 = h - w_0 \tag{10.38}$$

in which h is any non-zero integer to be determined.

Substituting (10.38) in (10.1) and simplifying, we have

$$h = x_0 + y_0 + 2z_0 + 2w_0$$

Thus, the second solution is given in the matrix form as

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \\ w_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 2 & 2 \\ 1 & 0 & 2 & 2 \\ 1 & 1 & 3 & 2 \\ 1 & 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \\ z_0 \\ w_0 \end{pmatrix}$$

Repeating the above process, the general solution to (4.1) in the matrix form as

$$\begin{pmatrix} x_n \\ y_n \\ z_n \\ w_n \end{pmatrix} = \begin{pmatrix} \frac{\tilde{y}_n + 3(-1)^n}{4} & \frac{\tilde{y}_n - (-1)^n}{4} & \tilde{x}_n & \frac{\tilde{y}_n - (-1)^n}{2} \\ \frac{\tilde{y}_n - (-1)^n}{4} & \frac{\tilde{y}_n + 3(-1)^n}{4} & \tilde{x}_n & \frac{\tilde{y}_n - (-1)^n}{2} \\ \frac{\tilde{x}_n}{2} & \frac{\tilde{x}_n}{2} & \tilde{y}_n & \tilde{x}_n \\ \frac{\tilde{y}_n - (-1)^n}{4} & \frac{\tilde{y}_n - (-1)^n}{4} & \tilde{x}_n & \frac{\tilde{y}_n + (-1)^n}{2} \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \\ z_0 \\ w_0 \end{pmatrix} \quad , n = 1,2,3,...$$

where $(\tilde{x}_n, \tilde{y}_n)$ is the general solution of $y^2 = 2x^2 + 1$ given by

$$\tilde{y}_n = \frac{1}{2}\left[\left(3 + 2\sqrt{2}\right)^{n+1} + \left(3 - 2\sqrt{2}\right)^{n+1}\right]$$

$$\tilde{x}_n = \frac{1}{2\sqrt{2}}\left[\left(3 + 2\sqrt{2}\right)^{n+1} - \left(3 - 2\sqrt{2}\right)^{n+1}\right], \quad n = 0, 1, 2, ...$$

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Chapter 11

A scrutiny of integer solutions to homogeneous quinary quadratic equation

11.1 Method of Analysis:

The second degree Diophantine equation with five unknowns to be solved is

$$4w^2 - x^2 - y^2 + z^2 = 16t^2 \tag{11.1}$$

The process of obtaining different sets of non-zero distinct integer solutions to (11.1) is exhibited below:

Set 1

The substitution of the linear transformations

$$x = 4P + 12Q, y = 8Y, z = 4(P - Q), w = 4(P + Q), t = 2T \tag{11.2}$$

in (11.1) leads to the space Pythagorean equation

$$P^2 = Q^2 + Y^2 + T^2 \tag{11.3}$$

which is satisfied by

$$P = a^2 + b^2 + c^2, T = a^2 - b^2 - c^2, Q = 2ab, Y = 2ac \tag{11.4}$$

In view of (11.2), one has the integer solutions to (11.1) given by

$$x = 4(a^2 + b^2 + c^2 + 6ab),\ y = 16ac,\ z = 4(a^2 + b^2 + c^2 - 2ab),$$
$$w = 4(a^2 + b^2 + c^2 + 2ab),\ t = 2(a^2 - b^2 - c^2)$$

Set 2

Introducing the linear transformations

$$x=(8a^2-1)s, y=4aY, z=s, w=4a^2s, t=aT \tag{11.5}$$

in (11.1), it simplifies to the Pythagorean equation

$$s^2=Y^2+T^2 \tag{11.6}$$

whose solutions may be taken as

$$s=p^2+q^2, T=p^2-q^2, Y=2pq \tag{11.7}$$

In view of (11.5), the integer solutions to (11.1) are given by

$$x=(8a^2-1)(p^2+q^2), y=8a\,pq, z=(p^2+q^2), w=4a^2(p^2+q^2)\ , t=a(p^2-q^2)$$

Note 1

The solutions to (2.146) is also taken as

$$s=p^2+q^2, Y=p^2-q^2, T=2pq$$

In this case, the integer solutions to (11.1) are given by

$$x=(8a^2-1)(p^2+q^2), y=4a(p^2-q^2), z=(p^2+q^2), w=4a^2(p^2+q^2)\ , t=2a\,pq$$

Set 3

Taking

$$x=4(P+Q), y=4(P-Q), w=4P, z=4Q \tag{11.8}$$

in (11.1), it reduces to

$$Q^2+t^2=2P^2 \tag{11.9}$$

After some algebra, it is seen that (11.9) is satisfied by

$$t = a^2 - b^2 + 2ab,$$
$$Q = a^2 - b^2 - 2ab,$$
$$P = a^2 + b^2$$

In view of (11.8), it is seen that

$$x = 8a(a-b),$$
$$y = 8b(a+b),$$
$$z = 4(a^2 - b^2 - 2ab),$$
$$w = 4(a^2 + b^2)$$

Thus, the above values of x, y, z, w, t satisfies (11.1).

Note 2

After performing a few calculations, (11.9) is also satisfied by

$$t = 2a^2 - b^2,$$
$$Q = 2a^2 + b^2 + 4ab,$$
$$P = 2a^2 + b^2 + 2ab$$

In view of (11.8), the corresponding values of x, y, z, w are found to be

$$x = 4(4a^2 + 2b^2 + 6ab),$$
$$y = -8ab,$$
$$z = 4(2a^2 + b^2 + 4ab),$$
$$w = 4(2a^2 + b^2 + 2ab)$$

Set 4

The choice

$$z = x + 4t \tag{11.10}$$

in (11.1) leads to

$$y^2 - 4w^2 = 8xt \tag{11.11}$$

which is expressed as the system of double equations as shown in Table 11.1 below:

Table 11.1: System of double equations

System	**1**	**2**	**3**	**4**
$y + 2w$	$8x$	$4x$	$8t$	$2x$
$y - 2w$	t	$2t$	x	$4t$

Solving each of the above systems, one obtains the values of x, y, w, t. In view of (11.10), the corresponding value of z is obtained. For simplicity, the integer solutions to the corresponding system of equations are exhibited below:

Solutions to system 1

$$x = s, y = 4s + 2k, z = s + 16k, w = 2s - k, t = 4k$$

Solutions to system 2

$$x = s, y = 2s + 2k, z = s + 8k, w = s - k, t = 2k$$

Solutions to system 3

$$x = 4s, y = 2s + 4k, z = 4s + 4k, w = 2k - s, t = k$$

Solutions to system 4

$$x = 2s, y = 2s + 2k, z = 2s + 4k, w = s - k, t = k$$

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Chapter 12

A portrayal of integer solutions to homogeneous quinary quadratic equation

12.1 Method of Analysis

The homogeneous quinary quadratic diophantine equation to be solved is

$$xy + XY = (k^2 + 2k - 1)w^2 \quad . \tag{12.1}$$

The process of obtaining different sets of non-zero distinct integer solutions to (12.1) is illustrated below.

Set12.1

Introduction of the linear transformations

$$x = (k+1)U - w, y = (k+1)U + w, X = (k+1)V - w, Y = (k+1)V + w \tag{12.2}$$

in (12.1) leads to the well-known Pythagorean equation

$$U^2 + V^2 = w^2 \quad , \tag{12.3}$$

whose solutions may be taken as

$$U = 2pq, V = p^2 - q^2 \tag{12.4}$$

and

$$w = p^2 + q^2 \quad . \tag{12.5}$$

Using (12.4) & (12.5) in (12.2) ,one has

$$\left.\begin{array}{l} x = 2(k+1)pq - (p^2 + q^2), y = 2(k+1)pq + (p^2 + q^2), \\ X = (k+1)(p^2 - q^2) - (p^2 + q^2), Y = (k+1)(p^2 - q^2) + (p^2 + q^2) \end{array}\right\}. \tag{12.6}$$

Thus,(12.5) and (12.6) give the required integer solutions to (12.1).

Set 12.2

Introduction of the linear transformations

$$x = u + v , y = u - v , X = v + s , Y = v - s \quad , \tag{12.7}$$

in (12.1) leads to the ternary quadratic diophantine equation

$$u^2 - s^2 = (k^2 + 2k - 1) w^2 \quad , \tag{12.8}$$

which can be written in the form of ratios as

$$\frac{u + s}{(k^2 + 2k - 1) w} = \frac{w}{u - s} = \frac{P}{Q} , Q \neq 0 \quad . \tag{12.9}$$

Solving the above system of double equations (12.9) , it is seen that

$$u = (k^2 + 2k - 1) P^2 + Q^2 , s = (k^2 + 2k - 1) P^2 - Q^2 \quad , \tag{12.10}$$

and

$$w = 2PQ \quad . \tag{12.11}$$

Using (12.10) in (12.7) ,we have

$$\left. \begin{array}{l} x = (k^2 + 2k - 1) P^2 + Q^2 + v , y = (k^2 + 2k - 1) P^2 + Q^2 - v , \\ X = v + (k^2 + 2k - 1) P^2 - Q^2 , Y = v - (k^2 + 2k - 1) P^2 + Q^2 \end{array} \right\} \tag{12.12}$$

Thus,(12.11) and (12.12) give the required integer solutions to (12.1).

Observation 12.1

Apart from (12.9) , (12.8) may be considered in the form of ratios as

$$\frac{u + s}{w} = \frac{(k^2 + 2k - 1) w}{u - s} = \frac{P}{Q} , Q \neq 0$$

The repetition of the above process leads to a different set of integer solutions to (12.1).

Set 12.3

Introduction of the linear transformations

$$x = (k + 1) w , y = (k - 1) w \, . \tag{12.13}$$

in (12.1) leads to the homogeneous ternary quadratic equation

$$XY = 2kw^2 \quad . \tag{12.14}$$

On considering different choices of factorization in (12.14) , the respective sets of integer solutions to (12.1) are given by

Set 12.3.1 $x = (k+1)w,\ y = (k-1)w,\ X = 2w,\ Y = kw$,

Set 12.3.2 $x = (k+1)w,\ y = (k-1)w,\ X = kw,\ Y = 2w$,

Set 12.3.3 $x = (k+1)w,\ y = (k-1)w,\ X = w^2,\ Y = 2k$,

Set 12.3.4 $x = (k+1)w,\ y = (k-1)w,\ X = 2w^2,\ Y = k$,

Set12.3.5 $x = (k+1)w,\ y = (k-1)w,\ X = 2wk,\ Y = w$,

Set 12.3.6 $x = (k+1)w,\ y = (k-1)w,\ X = w^2 k,\ Y = 2$,

Set 12.3.7 $x = (k+1)w,\ y = (k-1)w,\ X = 2w^2 k,\ Y = 1$.

Set12.4

Introduction of the linear transformations

$$x = (k+1)X,\ y = (k-1)Y,\ k \neq 1,2 \tag{12.15}$$

in (12.1) leads to the homogeneous ternary quadratic equation

$$k^2 XY = (k^2 + 2k - 1)w^2 \tag{12.16}$$

On considering different choices of factorization in (12.16) , the respective sets of integer solutions to (12.1) are given by

Set 12.4.1 $\begin{array}{l} x = (k+1)\ (k^2 + 2k - 1)^{2s}\ \alpha,\ y = (k-1)\ (k^2 + 2k - 1)\ \alpha, \\ X = (k^2 + 2k - 1)^{2s}\ \alpha,\ Y = (k^2 + 2k - 1)\ \alpha,\ w = k(k^2 + 2k - 1)^{s}\ \alpha. \end{array}$

Set 12.4.2 $\begin{array}{l} x = (k+1)\ (k^2 + 2k - 1)^{2s+1}\ \alpha,\ y = (k-1)\ \alpha, \\ X = (k^2 + 2k - 1)^{2s+1}\ \alpha,\ Y = \alpha,\ w = k(k^2 + 2k - 1)^{s}\ \alpha. \end{array}$

Set 12.4.3 $\begin{array}{l} x = (k+1)\,(k^2+2k-1)^{s+1}\,\alpha,\ y = (k-1)\,(k^2+2k-1)^{s}\,\alpha, \\ X = (k^2+2k-1)^{s+1}\,\alpha,\ Y = (k^2+2k-1)^{s}\,\alpha,\ w = k\,(k^2+2k-1)^{s}\,\alpha. \end{array}$

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Chapter 13

Designs of integer solutions to homogeneous quinary quadratic equation

13.1 Method of Analysis:

The second degree diophantine equation with five unknowns to be solved is

$$x^2 + y^2 + 4(z^2 + w^2) = 24t^2 \tag{13.1}$$

The process of obtaining different sets of non-zero distinct integer solutions to (1) is exhibited below:

Set 1:

The substitution of the linear transformations

$$x = 4t, y = 2t \tag{13.2}$$

in (13.1) leads to the pythagorean equation

$$t^2 = z^2 + w^2 \tag{13.3}$$

which is satisfied by

$$w = a^2 - b^2, z = 2ab, t = a^2 + b^2 \tag{13.4}$$

In view of (13.2), one has

$$\begin{aligned} x &= 4(a^2 + b^2), \\ y &= 2(a^2 + b^2) \end{aligned} \tag{13.5}$$

Thus,(13.4) and (13.5) represent the integer solutions to (13.1).

Set 2:

Introducing the linear transformations

$$x = 4u, y = 4v, z = u + v, w = u - v \tag{13.6}$$

in (13.1), it simplifies to the Pythagorean equation

$$t^2 = u^2 + v^2 \tag{13.7}$$

whose solutions may be taken as

$$t = p^2 + q^2, u = p^2 - q^2, v = 2pq \tag{13.8}$$

In view of (13.6), the integer solutions to (13.1) are given by

$$x = 4(p^2 - q^2), y = 8pq, z = (p^2 - q^2 + 2pq), w = (p^2 - q^2 - 2pq), t = (p^2 + q^2)$$

Set 3:

Taking

$$x = 4t, y = 2(z - 2\alpha), w = 2\alpha \tag{13.9}$$

in (13.1), it reduces to

$$z^2 - 2\alpha z + 4\alpha^2 - t^2 = 0 \tag{13.10}$$

Treating (13.10) as a quadratic in z and solving for z, it is seen that (13.10) is satisfied by

$$\begin{aligned} t &= 3r^2 + s^2, \\ .\alpha &= 2rs, \\ z &= 2rs \pm (3r^2 - s^2) \end{aligned}$$

n view of (13.9), it is seen that the corresponding values of x, y, w satisfying (13.1) are

.

$$\begin{aligned} x &= 4(3r^2 + s^2) \\ y &= -4rs \pm 2(3r^2 - s^2), \\ w &= 4rs \end{aligned}$$

Set 4:

Taking

$$x = 4(z + w), y = 2Y, t = z + w \tag{13.11}$$

in (13.1), it reduces to

$$z^2 + 4zw + w^2 - Y^2 = 0 \quad (13.12)$$

Treating (13.12) as a quadratic in z and solving for z, it is seen that (13.12) is satisfied by

$$Y = 3r^2 - s^2,$$
$$w = 2rs,$$
$$z = -4rs \pm (3r^2 + s^2).$$

In view of (13.11), it is seen that the corresponding values of x, y, t satisfying (13.1) are

$$x = -8rs \pm 4(3r^2 + s^2)$$
$$y = 2(3r^2 - s^2),$$
$$t = -2rs \pm (3r^2 + s^2)$$

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Chapter14

A classification of integer solutions to quinary quadratic equation

14.1 Method of analysis

The homogeneous quadratic equation with five unknowns under consideration is

$$x^2 + y^2 - 2zw = 2(c^2 + d^2)\,t^2 \qquad (14.1)$$

The substitution of the transformations

$$x = u + v, y = u - v, z = v + p, w = v - p, u \neq v, v \neq p \qquad (14.2)$$

in (14.1) leads to

$$u^2 + p^2 = (c^2 + d^2)\,t^2 \qquad (14.3)$$

Case 1:

Choose the values of $c\,,d$ such that $c^2 + d^2$ is square-free. Three patterns of integer solutions to (14.1) are studied.

Pattern 1

Assume

$$t = a^2 + b^2 \qquad (14.4)$$

Using (14.4) in (14.3) and employing factorization, consider

$$u + ip = (c + id)\,(a + ib)^2$$

giving

$$u = c\,(a^2 - b^2) - 2dab,$$
$$p = d\,(a^2 - b^2) + 2cab.$$

In view of (14.2) ,we have

$$\begin{aligned}
x &= c\,(a^2-b^2)-2dab+v,\\
y &= c\,(a^2-b^2)-2dab-v,\\
z &= v+d\,(a^2-b^2)+2cab,\\
w &= v-d\,(a^2-b^2)-2cab.
\end{aligned} \tag{14.5}$$

Thus , (14.4) & (14.5) satisfy (14.1).

Remarkable observation

It is worth to observe that ,for suitable values of a, b, c, d ,the numerical relation for second order Ramanujan numbers is represented by (14.3). A few examples are given below:

Example 1

Let $c = 2, d = 1, a = 2, b = 1$

Then ,we obtain $t = 5, u = 2, p = 11$

From (3) ,observe that

$$2^2 + 11^2 = 10^2 + 5^2 = 125$$

125 is the second order Ramanujan number as it is written as the sum of two squares in two different ways.

Example 2

Let $c = 3, d = 2, a = 2, b = 1$

Then ,we obtain $t = 5, u = 1, p = 18$

From (3) ,observe that

$$1^2 + 18^2 = 15^2 + 10^2 = 325$$

325 is the second order Ramanujan number as it is written as the sum of two squares in two different ways.

Example 3

Let $c = 4, d = -2, a = 2, b = -5$

Then ,we obtain $t = 29, u = -124, p = -38$

From (3) ,observe that

16820 is the second order Ramanujan number as it is written as the sum of two squares in two different ways.

Pattern 2

It is worth to be noted that the integer c^2+d^2 can be expressed as the product of complex conjugates as exhibited below :

$$c^2+d^2=\frac{(f+ig)(f-ig)}{(p^2+q^2)^2}$$

where

$$f=[c(p^2-q^2)+d(2pq)],$$
$$g=[c(2pq)-d(p^2-q^2)], p\neq q\neq 0$$

Following the procedure as given in Pattern 1, the integer solutions to (1) are obtained. For the benefit of the readers , an illustration is presented below:

Take

$$p=2, q=3, \ c=1, d=2$$

Now

$$f=1(-5)+2(12)=19 \ , g=1(12)-2(-5)=22$$

Therefore ,

$$c^2+d^2=5=\frac{(19+i22)\ (19-i22)}{13^2} \tag{14.6}$$

Using (14.4) & (14.6) in (14.3) and employing factorization , consider

$$u+ip=\frac{(19+i22)}{13}(a+ib)^2=\frac{(19+i22)}{13}[a^2-b^2+i2ab]$$

On equating the coefficients of corresponding terms , we have

$$u=\frac{1}{13}\{19[a^2-b^2]-44ab\},$$
$$p=\frac{1}{13}\{22[a^2-b^2]+38ab\} \ .$$

As the main thrust is to find integer solutions, replacing a by $13A$ and b by $13B$ in the above equations and from (14.2) ,the integer solutions to (14.1) are given by

$$x = 13\{19[A^2 - B^2] - 44AB\} + v,$$
$$y = 13\{19[A^2 - B^2] - 44AB\} + v,$$
$$z = v + 13\{22[A^2 - B^2] + 38AB\},$$
$$w = v - 13\{22[A^2 - B^2] + 38AB\},$$
$$t = 13^2[A^2 + B^2].$$

Pattern 3

Express (3) in the ratio form as

$$\frac{u - ct}{dt - p} = \frac{dt + p}{u + ct} = \frac{\alpha}{\beta}, \beta \neq 0$$

Solving the above system of double equations through the method of cross-multiplication and using (14.2) , the corresponding integer solutions to (14.1) are given by

$$x = 2\alpha\beta d - c(\alpha^2 - \beta^2) + v,$$
$$y = 2\alpha\beta d - c(\alpha^2 - \beta^2) - v,$$
$$z = v + 2\alpha\beta c + d(\alpha^2 - \beta^2),$$
$$w = v - 2\alpha\beta c - d(\alpha^2 - \beta^2),$$
$$t = (\alpha^2 + \beta^2).$$

Case 2

Choose the values of c, d such that $c^2 + d^2$ is a perfect square.

Consider c, d to be the legs of Pythagorean triangle. In otherwords , take $c = \alpha^2 - \beta^2, d = 2\alpha\beta, \alpha > \beta > 0$ so that $c^2 + d^2 = (\alpha + \beta)^2$.

The option

$$u = (\alpha + \beta)U, p = (\alpha + \beta)P \tag{14.6}$$

in (14.3) gives

$$U^2 + P^2 = t^2 \tag{14.7}$$

which is Pythagorean equation satisfied by

$$U = 2rs, P = r^2 - s^2, t = r^2 + s^2, r > s > 0 \tag{14.8}$$

From (14.6) & (14.2) , the integer solutions to (14.1) are given by

$$
\begin{aligned}
x &= 2rs(\alpha+\beta)+v, \\
y &= 2rs(\alpha+\beta)-v, \\
z &= v+(\alpha+\beta)(r^2-s^2), \\
w &= v-(\alpha+\beta)(r^2-s^2)
\end{aligned}
$$

jointly with t given by (14.8).

Conclusions

It is hoped that these problems may create an interest in the hearts of researchers and lovers of mathematics who approach it with pure love for its own beauty. The authors hope that, seeing the excitement of solving this multiple variables quadratic Diophantine equations, young mathematicians and researchers realize that there are lots and lots of other problems in Number Theory which are going to be challenging in the future. There is no wonder that Diophantine equations are beautiful and tricky enough to keep a mathematician occupied for entire life.

It is worth to observe that Number Theory distinguishes itself through its intrinsic beauty, offering both enjoyment and excitement. The outstanding results in this study of Diophantine equation will be useful for all readers.

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