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Preface

One of the areas of Number theory that has attracted many mathematicians since antiquity is the subject of diophantine equations. A diophantine equation is a polynomial equation in two or more unknowns such that only the integer solutions are determined. No doubt that diophantine equation possess supreme beauty and it is the most powerful creation of the human spirit. A pell equation is a type of non-linear diophantine equation in the form $y^2 - Dx^2 = \pm 1$ where $D > 0$ and square-free. The above equation is also called the Pell-Fermat equation. In Cartesian co-ordinates, this equation has the form of a hyperbola. The binary quadratic diophantine equation having the form

$$y^2 = Dx^2 + N \quad (N > 0\,;\, D > 0\text{, a non-square integer}) \tag{1}$$

is referred to as the positive form of the pell equation and the form

$$y^2 = Dx^2 - N \quad (N > 0\,;\, D > 0\text{, non-square integer}) \tag{2}$$

is called the negative form of the pell equation or related pell equation. It is worth to remind that (2) is solvable for only certain values of D and always in the case of (1). An obvious generalisation to the Pell equation is the equation of the form $ax^2 - by^2 = N$; $a, b > 0$, $N \neq 0$ which is known as Pell-like equation.

Pell equations arise in the investigation of numbers which are figurate in more than one way, for example, simultaneously square & triangular and as such they are extremely important in Number theory. In the solution of cubic equation and in certain other situations it is desirable to have a method for extracting the cube root of a binomial surd. This may be accomplished by the aid of the pell equation. We use pell equation to solve Archimedes' Cattle problem. Pell's equation is connected to algebraic number theory, Chebyshev polynomials and continued fractions. Other applications include solving problems involving double equations, rational approximations to square roots, sums of consecutive integers, Pythagorean triangles with consecutive legs, consecutive Heronian triangles, sums of n and $n+1$ consecutive squares and so on. Man's love for numbers is perhaps older than number theory. The love for large numbers may be a motivation for pellian equation.

In studies on Diophantine equations of degree two with two and three unknowns, significant success was attained only in the twentieth century. There has been interest in determining all solutions in integers to quadratic Diophantine equations among mathema6ticians.

The main thrust in this book is on solving second degree Diophantine equations with two and three variables. This book contains a reasonable collection of special quadratic Diophantine problems in two and three variables distributed in 12 chapters. The process of getting different sets of integer solutions to each of the quadratic Diophantine equations considered in this book are illustrated in an elegant manner. The articles with solutions and properties presented in chapters 1, 2 & 3 are Pell equations and in chapters 4,5 &6 are Pell-like equations. The articles with solutions presented in chapters 7-12 are quadratic equations with three unknowns of the form $x^2 + y^2 = (a^2 + b^2)z^2$. In Cartesian co-ordinates, this equation has the form of a right circular cone.

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Chapter 1

Integer Solutions of The Positive Pell Equation $y^2 = 3x^2 + \alpha^2 + 2\alpha - 2$

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Abstract: A binary quadratic equation of the form $y^2 = Dx^2 + 1$, where D is non-square positive integer has been study by various mathematicians for it non-trivial integral solutions when D takes different integral values. The binary quadratic Diophantine equation represented by the positive pellian $y2 = 3x^2 + \alpha^2 + 2\alpha - 2$ is analysed for its non-zero distinct solutions. A few interesting relations among the solutions are given. Further, employing the solutions of the above hyperbola, the solutions of other choices of hyperbolas and parabolas are obtained.

Keywords: Binary quadratic, Hyperbola, Parabola, Pell equation, Integral solutions.
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1.1 Introduction

A binary quadratic equation of the form $y^2 = Dx^2 + 1$, where D is non-square positive integer has been study by various mathematicians for it non-trivial integral solutions when D takes different integral values (Carmichael., 1959; Dickson., 1952; Mordell., 1969). For an extensive review of various problems, one may refer (Gopalan et.al., 2012; Mahalakshmi, Shanthi .,2023; Shanthi, Mahalakshmi .,2023).In this communication, yet another interesting hyperbola given by $y^2 = 3x^2 + \alpha^2 + 2\alpha - 2$ is considered and infinitely many integer solutions are obtained. A few interesting properties among the solutions are obtained. Further, employing the solutions of the above hyperbola, we have obtained solutions of other choices of hyperbola and parabola.

1.2Method of analysis:

The Positive Pell equation representing hyperbola under consideration is

$$y^2 = 3x^2 + \alpha^2 + 2\alpha - 2 \qquad (1.1)$$

whose smallest positive integer solution is

$$x_0 = 1, y_0 = \alpha + 1$$

To obtain the other solutions of (1.1), consider the Pell equation

$$y^2 = 3x^2 + 1$$

whose general solution is given by

$$\tilde{x}_n = \frac{1}{2\sqrt{3}} g_n; \tilde{y}_n = \frac{1}{2} f_n$$

where

$$f_n = (2+\sqrt{3})^{n+1} + (2-\sqrt{3})^{n+1}$$

$$g_n = (2+\sqrt{3})^{n+1} - (2-\sqrt{3})^{n+1}$$

Applying Brahmagupta lemma between (x_0, y_0)&$(\tilde{x}_n, \tilde{y}_n)$the other integer solution of (1.1) are given by

$$x_{n+1} = \frac{3f_n}{6} + \frac{(\alpha+1)\sqrt{3}g_n}{6}$$

$$y_{n+1} = \frac{3(\alpha+1)f_n}{6} + \frac{3\sqrt{3}g_n}{6}$$

The recurrence relations satisfied by x and y are given by

$$x_{n+1} - 4x_{n+2} + x_{n+3} = 0$$

$$y_{n+1} - 4y_{n+2} + y_{n+3} = 0$$

A few numerical examples are given in the following table 1.1

Table: 1.1 Numerical values

n	x_n	y_n
0	1	$\alpha + 1$
1	$\alpha + 3$	$2\alpha + 5$
2	$4\alpha + 11$	$7\alpha + 19$
3	$15\alpha + 41$	$26\alpha + 71$
4	$56\alpha + 153$	$97\alpha + 265$

From the above table we observe some interesting properties among the solutions which are presented below:

1.3 Relations between solutions

- $x_{n+1} - 4x_{n+2} + x_{n+3} = 0$
- $2x_{n+1} - x_{n+2} + y_{n+1} = 0$
- $x_{n+1} - 2x_{n+2} + y_{n+2} = 0$
- $2x_{n+1} - 7x_{n+2} + y_{n+3} = 0$
- $7x_{n+1} - x_{n+3} + 4y_{n+1} = 0$
- $x_{n+1} - x_{n+3} + 2y_{n+2} = 0$
- $x_{n+1} - 7x_{n+3} + 4y_{n+3} = 0$
- $3x_{n+1} + 2y_{n+1} - y_{n+2} = 0$
- $12x_{n+1} + 7y_{n+1} - y_{n+3} = 0$
- $3x_{n+1} + 7y_{n+2} - 2y_{n+3} = 0$
- $y_{n+1} + 7x_{n+2} - 2x_{n+3} = 0$
- $y_{n+2} + 2x_{n+2} - x_{n+3} = 0$
- $y_{n+3} + x_{n+2} - 2x_{n+3} = 0$
- $y_{n+1} + 3x_{n+2} - 2y_{n+2} = 0$
- $y_{n+1} + 6x_{n+2} - y_{n+3} = 0$
- $2y_{n+2} + 3x_{n+2} - y_{n+3} = 0$
- $7y_{n+2} - 2y_{n+1} - 3x_{n+3} = 0$
- $7y_{n+3} - y_{n+1} - 12x_{n+3} = 0$
- $2y_{n+3} - y_{n+2} - 3x_{n+3} = 0$
- $y_{n+3} - 4y_{n+2} + y_{n+1} = 0$

1.4 Each of the following expressions represents a cubical integers

- $\frac{1}{(\alpha^2+2\alpha-2)}\left[[(2\alpha+2)x_{3n+4} - (10+4\alpha)x_{3n+3}] + 3[(2\alpha+2)x_{n+2} - (10+4\alpha)x_{n+1}]\right]$

- $\frac{1}{2(\alpha^2+2\alpha-2)}\left[[(\alpha+1)x_{3n+5} - (19+7\alpha)x_{3n+3}] + 3[(\alpha+1)x_{n+3} - (19+7\alpha)x_{n+1}]\right]$

- $\frac{1}{(\alpha^2+2\alpha-2)}\left[[(2\alpha+2)y_{3n+3}-6x_{3n+3}]+3[(2\alpha+2)y_{n+1}-6x_{n+1}]\right]$

- $\frac{1}{(\alpha^2+2\alpha-2)}\left[[(\alpha+1)y_{3n+4}-(9+3\alpha)x_{3n+3}]+3[(\alpha+1)y_{n+2}-(9+3\alpha)x_{n+1}]\right]$

- $\frac{1}{7(\alpha^2+2\alpha-2)}\left[2[(\alpha+1)y_{3n+5}-(33+12\alpha)x_{3n+3}]+6[(\alpha+1)y_{n+3}-(33+12\alpha)x_{n+1}]\right]$

- $\frac{1}{(\alpha^2+2\alpha-2)}\left[2[(2\alpha+5)x_{3n+5}-(19+7\alpha)x_{3n+4}]+6[(2\alpha+5)x_{n+3}-(19+7\alpha)x_{n+1}]\right]$

- $\frac{1}{(\alpha^2+2\alpha-2)}\left[[(2\alpha+5)y_{3n+3}-3x_{3n+4}]+3[(2\alpha+5)y_{n+1}-3x_{n+2}]\right]$

- $\frac{1}{(\alpha^2+2\alpha-2)}\left[[(4\alpha+10)y_{3n+4}-(18+6\alpha)x_{3n+4}]+3[(4\alpha+10)y_{n+2}-(18+6\alpha)x_{n+2}]\right]$

- $\frac{1}{(\alpha^2+2\alpha-2)}\left[[(2\alpha+5)y_{3n+5}-(33+12\alpha)x_{3n+4}]+3[(2\alpha+5)y_{n+3}-(33+12\alpha)x_{n+2}]\right]$

- $\frac{1}{7(\alpha^2+2\alpha-2)}\left[[(14\alpha+38)y_{3n+3}-6x_{3n+5}]+3[(14\alpha+38)y_{n+1}-6x_{n+3}]\right]$

- $\frac{1}{(\alpha^2+2\alpha-2)}\left[[(7\alpha+19)y_{3n+4}-(9+3\alpha)x_{3n+5}]+3[(7\alpha+19)y_{n+2}-(9+3\alpha)x_{n+3}]\right]$

- $\frac{1}{(\alpha^2+2\alpha-2)}\left[[(14\alpha+38)y_{3n+5}-(66+24\alpha)y_{3n+5}]+3[(14\alpha+38)y_{n+3}-(66+24\alpha)x_{n+3}]\right]$

- $\frac{1}{(\alpha^2+2\alpha-2)}\left[[(2\alpha+6)y_{3n+3}-2y_{3n+4}]+3[(2\alpha+6)y_{n+1}-2y_{n+2}]\right]$

- $\frac{1}{2(\alpha^2+2\alpha-2)}[[(4\alpha+11)y_{3n+3}-y_{3n+5}]+3[(4\alpha+11)y_{n+1}-y_{n+3}]]$

- $\frac{1}{(\alpha^2+2\alpha-2)}[[(8\alpha+22)y_{3n+4}-(6+2\alpha)y_{3n+5}]+3[(8\alpha+22)y_{n+2}-(6+2\alpha)y_{n+3}]]$

1.5 Each of the following expressions represents a Bi-quadratic Integer

- $\frac{1}{(\alpha^2+2\alpha-2)}[(2\alpha+2)x_{4n+5}-(10+4\alpha)x_{4n+4}+4[(2\alpha+2)x_{2n+3}-(10+4\alpha)x_{2n+2}]]+6$

- $\frac{1}{2(\alpha^2+2\alpha-2)}[(\alpha+1)x_{4n+6}-(19+7\alpha)x_{4n+4}+4[(\alpha+1)x_{2n+4}-(19+7\alpha)x_{2n+2}]]+6$

- $\frac{1}{(\alpha^2+2\alpha-2)}[(2\alpha+2)y_{4n+4}-6x_{4n+4}+4[(2\alpha+2)y_{2n+2}-6x_{2n+2}]]+6$

- $\frac{1}{(\alpha^2+2\alpha-2)}[(\alpha+1)y_{4n+5}-(9+3\alpha)x_{4n+4}+4[(\alpha+1)y_{2n+3}-(9+3\alpha)x_{2n+2}]]+6$

- $\frac{1}{7(\alpha^2+2\alpha-2)}[(2\alpha+2)y_{4n+6}-(66+24\alpha)x_{4n+4}+8[(\alpha+1)y_{2n+4}-(33+12\alpha)x_{2n+2}]]+6$

- $\frac{1}{(\alpha^2+2\alpha-2)}[(4\alpha+10)x_{4n+6}-(95+35\alpha)x_{4n+5}+8[(2\alpha+5)x_{2n+4}-(19+7\alpha)x_{2n+3}]]+6$

- $\frac{1}{(\alpha^2+2\alpha-2)}[(2\alpha+5)x_{4n+4}-3x_{4n+5}+4[(2\alpha+5)y_{2n+2}-3x_{2n+3}]]+6$

- $\frac{1}{(\alpha^2+2\alpha-2)}[(4\alpha+10)y_{4n+5}-(18+6\alpha)x_{4n+5}+4[(4\alpha+10)y_{2n+3}-(18+6\alpha)x_{2n+3}]]+6$

- $\frac{1}{(\alpha^2+2\alpha-2)}\left[(2\alpha+5)y_{4n+6}-(33+12\alpha)x_{4n+5}+4[(2\alpha+5)y_{2n+4}-(33+12\alpha)x_{2n+3}]\right]+6$

- $\frac{1}{7(\alpha^2+2\alpha-2)}\left[(14\alpha+38)y_{4n+4}-6x_{4n+6}+4[(14\alpha+38)y_{2n+2}-6x_{2n+4}]\right]+6$

- $\frac{1}{(\alpha^2+2\alpha-2)}\left[(7\alpha+19)y_{4n+5}-(9+3\alpha)x_{4n+6}+4[(7\alpha+19)y_{2n+3}-(9+3\alpha)x_{2n+4}]\right]+6$

- $\frac{1}{(\alpha^2+2\alpha-2)}\left[(14\alpha+38)y_{4n+6}-(66+24\alpha)x_{4n+6}+4[(14\alpha+38)y_{2n+4}-(66+24\alpha)x_{2n+4}]\right]+6$

- $\frac{1}{(\alpha^2+2\alpha-2)}\left[(2\alpha+6)y_{4n+4}-2y_{4n+5}+4[(2\alpha+6)y_{2n+2}-2y_{2n+3}]\right]+6$

- $\frac{1}{2(\alpha^2+2\alpha-2)}\left[(4\alpha+11)y_{4n+4}-y_{4n+6}+4[(4\alpha+11)y_{2n+2}-y_{2n+4}]\right]+6$

- $\frac{1}{(\alpha^2+2\alpha-2)}\left[(8\alpha+22)y_{4n+5}-(6+2\alpha)y_{4n+6}+4[(8\alpha+22)y_{2n+3}-(6+2\alpha)y_{2n+4}]\right]+6$

1.6 Each of the following expressions represents a Quintic Integer

- $\frac{1}{(\alpha^2+2\alpha-2)}\left[(2\alpha+2)x_{5n+6}-(10+4\alpha)x_{5n+5}+5[(2\alpha+2)x_{3n+4}-(10+4\alpha)x_{3n+3}]+10[(2\alpha+2)x_{n+2}-(10+4\alpha)x_{n+1}]\right]$

- $\frac{1}{2(\alpha^2+2\alpha-2)}\left[(\alpha+1)x_{5n+7}-(19+7\alpha)x_{5n+5}+5[(\alpha+1)x_{3n+5}-(19+7\alpha)x_{3n+3}]+10[(\alpha+1)x_{n+3}-(19+7\alpha)x_{n+1}]\right]$

- $\frac{1}{(\alpha^2+2\alpha-2)}\left[(2\alpha+2)y_{5n+5}-6x_{5n+5}+5[(2\alpha+2)y_{3n+3}-6x_{3n+3}]+10[(2\alpha+2)y_{n+1}-6x_{n+1}]\right]$

- $\frac{1}{(\alpha^2+2\alpha-2)}\left[(\alpha+1)y_{5n+6}-(9+3\alpha)x_{5n+5}+5[(\alpha+1)y_{3n+4}-(9+3\alpha)x_{3n+3}]+10[(\alpha+1)y_{n+2}-(9+3\alpha)x_{n+1}]\right]$

- $\frac{1}{7(\alpha^2+2\alpha-2)}\left[(2\alpha+2)y_{5n+7}-(66+24\alpha)x_{5n+5}+5[(2\alpha+2)y_{3n+5}-(66+24\alpha)x_{3n+3}]+10[(2\alpha+2)y_{n+3}-(66+24\alpha)x_{n+1}]\right]$

- $\frac{1}{(\alpha^2+2\alpha-2)}\left[(4\alpha+10)x_{5n+7}-(38+14\alpha)x_{5n+6}+5[(4\alpha+10)x_{3n+5}-(38+14\alpha)x_{3n+4}]+10[(4\alpha+10)x_{n+3}-(38+14\alpha)x_{n+2}]\right]$
- $\frac{1}{(\alpha^2+2\alpha-2)}\left[(2\alpha+5)y_{5n+5}-3x_{5n+6}+5[(2\alpha+5)y_{3n+3}-3x_{3n+4}]+10[(2\alpha+5)y_{n+1}-3x_{n+2}]\right]$

- $\frac{1}{(\alpha^2+2\alpha-2)}\left[(4\alpha+10)y_{5n+6}-(18+6\alpha)x_{5n+6}+5[(4\alpha+10)y_{3n+4}-(18+6\alpha)x_{3n+4}]+10[(4\alpha+10)y_{n+2}-(18+6\alpha)x_{n+2}]\right]$

- $\frac{1}{(\alpha^2+2\alpha-2)}\left[(2\alpha+5)y_{5n+7}-(33+12\alpha)x_{5n+6}+5[(2\alpha+5)y_{3n+5}-(33+12\alpha)x_{3n+4}]+10[(2\alpha+5)y_{n+3}-(33+12\alpha)x_{n+2}]\right]$

- $\frac{1}{7(\alpha^2+2\alpha-2)}\left[(14\alpha+38)y_{5n+5}-6x_{5n+7}+5[(14\alpha+38)y_{3n+3}-6x_{3n+5}]+10[(14\alpha+38)y_{n+1}-6x_{n+3}]\right]$

- $\frac{1}{(\alpha^2+2\alpha-2)}\left[(7\alpha+19)y_{5n+6}-(9+3\alpha)x_{5n+7}+5[(7\alpha+19)y_{3n+4}-(9+3\alpha)x_{3n+5}]+10[(7\alpha+19)y_{n+2}-(9+3\alpha)x_{n+3}]\right]$

- $\frac{1}{(\alpha^2+2\alpha-2)}\left[(14\alpha+38)y_{5n+7}-(66+24\alpha)x_{5n+7}+5[(14\alpha+38)y_{3n+5}-(66+24\alpha)x_{3n+5}]+10[(14\alpha+38)y_{n+3}-(66+24\alpha)x_{n+3}]\right]$

- $\frac{1}{(\alpha^2+2\alpha-2)}\left[(2\alpha+6)y_{5n+5}-2y_{5n+6}+5[(2\alpha+6)y_{3n+3}-2y_{3n+4}]+10[(2\alpha+6)y_{n+1}-2y_{n+2}]\right]$

- $\frac{1}{2(\alpha^2+2\alpha-2)}\left[(4\alpha+11)y_{5n+5}-y_{5n+7}+5[(4\alpha+11)y_{3n+3}-y_{3n+5}]+10[(4\alpha+11)y_{n+1}-y_{n+3}]\right]$

➢ $\frac{1}{(\alpha^2+2\alpha-2)}\left[(8\alpha+22)y_{5n+6}-(6+2\alpha)y_{5n+7}+5[(8\alpha+22)y_{3n+4}-(6+2\alpha)y_{3n+5}]+10[(8\alpha+22)y_{n+2}-(6+2\alpha)y_{n+3}]\right]$

1.7 Remarkable observations:

1. Employing linear combinations among the solutions of (1.1), one may generate integer solutions for other choices of hyperbola which are presented in table 1.2 below

Table: 1.2 Hyperbola

S.NO.	Hyperbola	(P,Q)
1.	$3P^2-Q^2=12(\alpha^2+2\alpha-2)^2$	$P=(2\alpha+2)x_{n+2}-(10+4\alpha)x_{n+1}$ $Q=(18+6\alpha)x_{n+1}-6x_{n+2}$
2.	$3P^2-Q^2=48(\alpha^2+2\alpha-2)^2$	$P=(\alpha+1)x_{n+3}-(19+7\alpha)x_{n+1}$ $Q=(33+12\alpha)x_{n+1}-3x_{n+3}$
3.	$3P^2-Q^2=12(\alpha^2+2\alpha-2)^2$	$P=(2\alpha+2)y_{n+1}-6x_{n+1}$ $Q=(6+6\alpha)x_{n+1}-6y_{n+1}$
4.	$P^2-3Q^2=144(\alpha^2+2\alpha-2)^2$	$P=(\alpha+1)y_{n+2}-(9+3\alpha)x_{n+1}$ $Q=(15+6\alpha)x_{n+1}-3y_{n+2}$
5.	$3P^2-Q^2=588(\alpha^2+2\alpha-2)^2$	$P=(2\alpha+2)y_{n+3}-(66+24\alpha)x_{n+1}$ $Q=(114+42\alpha)x_{n+1}-6y_{n+3}$
6.	$3P^2-Q^2=12(\alpha^2+2\alpha-2)^2$	$P=2(2\alpha+5)x_{n+3}-2(19+7\alpha)x_{n+2}$ $Q=6(11+4\alpha)x_{n+2}-6(\alpha+3)x_{n+3}$
7.	$3P^2-Q^2=12(\alpha^2+2\alpha-2)^2$	$P=(2\alpha+5)y_{n+1}-3x_{n+2}$ $Q=3(1+\alpha)x_{n+2}-3(\alpha+3)y_{n+1}$

8.	$3P^2 - Q^2 = 12(\alpha^2 + 2\alpha - 2)^2$	$P = (4\alpha + 10)y_{n+2} - (18 + 6\alpha)x_{n+2}$ $Q = (30 + 12\alpha)x_{n+2} - (18 + 6\alpha)y_{n+2}$
9.	$3P^2 - Q^2 = 12(\alpha^2 + 2\alpha - 2)^2$	$P = (2\alpha + 5)y_{n+3} - (33 + 12\alpha)x_{n+2}$ $Q = (57 + 21\alpha)x_{n+2} - (3\alpha + 9)y_{n+3}$
10.	$3P^2 - Q^2 = 588(\alpha^2 + 2\alpha - 2)^2$	$P = (14\alpha + 38)y_{n+1} - 6x_{n+3}$ $Q = (6\alpha + 6)x_{n+3} - (24\alpha + 66)y_{n+1}$
11.	$3P^2 - Q^2 = 12(\alpha^2 + 2\alpha - 2)^2$	$P = (7\alpha + 19)y_{n+2} - (9 + 3\alpha)x_{n+3}$ $Q = (15 + 6\alpha)x_{n+3} - (12\alpha + 33)y_{n+2}$
12.	$3P^2 - Q^2 = 12(\alpha^2 + 2\alpha - 2)^2$	$P = (14\alpha + 38)y_{n+3} - (66 + 24\alpha)x_{n+3}$ $Q = (114 + 42\alpha)x_{n+3} - (24\alpha + 66)y_{n+3}$
13.	$3P^2 - Q^2 = 12(\alpha^2 + 2\alpha - 2)^2$	$P = (2\alpha + 6)y_{n+1} - 2y_{n+2}$ $Q = (2 + 2\alpha)y_{n+2} - (10 + 4\alpha)y_{n+1}$
14.	$3P^2 - Q^2 = 48(\alpha^2 + 2\alpha - 2)^2$	$P = (4\alpha + 11)y_{n+1} - y_{n+3}$ $Q = (1 + \alpha)y_{n+3} - (7\alpha + 19)y_{n+1}$
15.	$3P^2 - Q^2 = 12(\alpha^2 + 2\alpha - 2)^2$	$P = (8\alpha + 22)y_{n+2} - (6 + 2\alpha)y_{n+3}$ $Q = (10 + 4\alpha)y_{n+3} - (14\alpha + 38)y_{n+2}$

2. Employing linear combinations among the solutions of (1.1), one may generate integer solutions for other choices of parabola which are presented in the Table 1.3 below

Table: 1.3 Parabola

S.No	**Parabola**	**(R,Q)**
1	$3R(\alpha^2 + 2\alpha - 2) - Q^2 = 12(\alpha^2 + 2\alpha - 2)^2$	$R = (2\alpha + 2)x_{2n+3} - (10 + 4\alpha)x_{2n+2} + 2(\alpha^2 + 2\alpha - 2)$ $Q = (18 + 6\alpha)x_{n+1} - 6x_{n+2}$

2	$6R(\alpha^2+2\alpha-2)-Q^2 = 48(\alpha^2+2\alpha-2)^2$	$R=(\alpha+1)x_{2n+4}-(19+7\alpha)x_{2n+2}+2(\alpha^2+2\alpha-2)$ $Q=(33+12\alpha)x_{n+1}-3x_{n+3}$
3	$3R(\alpha^2+2\alpha-2)-Q^2 = 12(\alpha^2+2\alpha-2)^2$	$R=(2\alpha+2)y_{2n+2}-6x_{2n+2}+2(\alpha^2+2\alpha-2)$ $Q=(6+6\alpha)x_{n+1}-6y_{n+1}$
4	$12R(\alpha^2+2\alpha-2)-Q^2 = 48(\alpha^2+2\alpha-2)^2$	$R=(\alpha+1)y_{2n+3}-(9+3\alpha)x_{2n+2}+2(\alpha^2+2\alpha-2)$ $Q=(15+6\alpha)x_{n+1}-3y_{n+2}$
5	$21R(\alpha^2+2\alpha-2)-Q^2 = 588(\alpha^2+2\alpha-2)^2$	$R=2[(\alpha+1)y_{2n+4}-(33+12\alpha)x_{2n+2}]+14(\alpha^2+2\alpha-2)$ $Q=(114+42\alpha)x_{n+1}-6y_{n+3}$
6	$3R(\alpha^2+2\alpha-2)-Q^2 = 12(\alpha^2+2\alpha-2)^2$	$R=2[(5+2\alpha)x_{2n+4}-(19+7\alpha)x_{2n+3}]+2(\alpha^2+2\alpha-2)$ $Q=6(11+4\alpha)x_{n+2}-6(\alpha+3)x_{n+3}$
7	$3R(\alpha^2+2\alpha-2)-Q^2 = 12(\alpha^2+2\alpha-2)^2$	$R=[(5+2\alpha)y_{2n+2}-3(x_{2n+3})]+2(\alpha^2+2\alpha-2)$ $Q=3(1+\alpha)x_{n+2}-3(\alpha+3)y_{n+1}$
8	$3R(\alpha^2+2\alpha-2)-Q^2 = 12(\alpha^2+2\alpha-2)^2$	$R=[(10+4\alpha)y_{2n+3}-(6\alpha+18)x_{2n+3}]+2(\alpha^2+2\alpha-2)$ $Q=(30+12\alpha)x_{n+2}-(18+6\alpha)y_{n+2}$
9	$3R(\alpha^2+2\alpha-2)-Q^2 = 12(\alpha^2+2\alpha-2)^2$	$R=(5+2\alpha)y_{2n+4}-(33+12\alpha)x_{2n+3}+2(\alpha^2+2\alpha-2)$ $Q=(57+21\alpha)x_{n+2}-(3\alpha+9)y_{n+3}$
10	$21R(\alpha^2+2\alpha-2)-Q^2 = 588(\alpha^2+2\alpha-2)^2$	$R=(38+14\alpha)y_{2n+2}-6x_{2n+4}+14(\alpha^2+2\alpha-2)$ $Q=(6\alpha+6)x_{n+3}-(24\alpha+66)y_{n+1}$
11	$3R(\alpha^2+2\alpha-2)-Q^2 = 12(\alpha^2+2\alpha-2)^2$	$R=(19+7\alpha)y_{2n+3}-(9+3\alpha)x_{2n+4}+2(\alpha^2+2\alpha-2)$ $Q=(15+6\alpha)x_{n+3}-(12\alpha+33)y_{n+2}$
12	$3R(\alpha^2+2\alpha-2)-Q^2 = 12(\alpha^2+2\alpha-2)^2$	$R=(38+14\alpha)y_{2n+4}-(66+24\alpha)x_{2n+4}+2(\alpha^2+2\alpha-2)$ $Q=(114+42\alpha)x_{n+3}-(66+24\alpha)y_{n+3}$
13	$3R(\alpha^2+2\alpha-2)-Q^2 = 12(\alpha^2+2\alpha-2)^2$	$R=(2\alpha+6)y_{2n+2}-2y_{2n+3}+2(\alpha^2+2\alpha-2)$ $Q=(2+2\alpha)y_{n+2}-(10+4\alpha)y_{n+1}$
14	$6R(\alpha^2+2\alpha-2)-Q^2 = 48(\alpha^2+2\alpha-2)^2$	$R=(11+4\alpha)y_{2n+2}-y_{2n+3}+4(\alpha^2+2\alpha-2)$ $Q=(1+\alpha)y_{n+3}-(7\alpha+19)y_{n+1}$

15	$3R(\alpha^2+2\alpha-2)-Q^2 = 12(\alpha^2+2\alpha-2)^2$	$R=(22+8\alpha)y_{2n+3}-(2\alpha+6)y_{2n+4}+2(\alpha^2+2\alpha-2)$ $Q=(10+4\alpha)y_{n+3}-(14\alpha+38)y_{n+2}$

1.8 Conclusion:

In this paper, we have presented infinitely many integer solutions for the Diophantine equations represented by the positive pell equation$y^2 = 3x^2+\alpha^2+2\alpha-2$. As the binary quadratic Diophantine equations are rich in variety, one may search for the other choices of pell equations and determine the solutions with the suitable properties.

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Chapter 2

A GlimpseOn Integer Solutions to Binary Equations $y^2 = 3x^2 + \alpha^2 + 6\alpha - 3$

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Abstract:A binary quadratic equation of the form $y^2 = Dx^2 + 1$, where D is non-square positive integer has been study by various mathematicians for it non-trivial integral solutions when D takes different integral values. The binary quadratic Diophantine equation represented by the positive pellian $y2_{\square} = 3x^2 + \alpha^2 + 6\alpha - 3$ is analysed for its non-zero distinct solutions. A few interesting relations among the solutions are given. Further,employing the solutions of the above hyperbola, the solutions of other choices of hyperbolas and parabolas are obtained.

Keywords: Binary quadratic, Hyperbola, Parabola, Pell equation, Integral solutions.
2010 Mathematics subject classification: 11D09

2.1 Introduction

A binary quadratic equation of the form $y^2 = Dx^2 + 1$, where D is non-square positive integer has been study by various mathematicians for it non-trivial integral solutions when D takes different integral values(Carmichael., 1959; Dickson., 1952; Mordell., 1969). For an extensive review of various problems, one may refer(Gopalan et.al., 2015; Mahalakshmi, Shanthi .,2023; Shanthi.,2023) In this communication, yet another interesting hyperbola given by $y^2 = 3x^2 + \alpha^2 + 2\alpha - 2$ is considered and infinitely many integer solutions are obtained. A few interesting properties among the solutions are obtained. Further, employing the solutions of the above hyperbola, we have obtained solutions of other choices of hyperbola and parabola.

2.2Method of analysis:

The Positive Pell equation representing hyperbola under consideration is

$$y^2 = 3x^2 + \alpha^2 + 6\alpha - 3 \qquad (2.1)$$

whose smallest positive integer solution is

$$x_0 = 2, y_0 = \alpha + 3$$

To obtain the other solutions of (2.1), consider the Pell equation

$$y^2 = 3x^2 + 1$$

whose general solution is given by

$$\tilde{x}_n = \frac{1}{2\sqrt{3}} g_n; \tilde{y}_n = \frac{1}{2} f_n$$

where

$$f_n = (2+\sqrt{3})^{n+1} + (2-\sqrt{3})^{n+1}$$

$$g_n = (2+\sqrt{3})^{n+1} - (2-\sqrt{3})^{n+1}$$

Applying Brahmagupta lemma between (x_0, y_0)&$(\tilde{x}_n, \tilde{y}_n)$ the other integer solution of (2.1) are given by

$$x_{n+1} = \frac{2\sqrt{3} f_n}{2\sqrt{3}} + \frac{(\alpha+3) g_n}{2\sqrt{3}}$$

$$y_{n+1} = \frac{(\alpha+3) f_n}{2} + \frac{2\sqrt{3} g_n}{2}$$

The recurrence relations satisfied by x and y are given by

$$x_{n+1} - 4x_{n+2} + x_{n+3} = 0$$

$$y_{n+1} - 4y_{n+2} + y_{n+3} = 0$$

A few numerical examples are given in the following table: 2.1

Table: 2.1 Numerical values

n	x_n	y_n
0	2	$\alpha+3$
1	$\alpha+7$	$2\alpha+12$
2	$4\alpha+26$	$7\alpha+45$
3	$15\alpha+97$	$26\alpha+168$
4	$56\alpha+362$	$97\alpha+627$

From the above table we observe some interesting properties among the solutions which are presented below:

2.3 Relations between solutions

- $x_{n+1}-4x_{n+2}+x_{n+3}=0$
- $2x_{n+1}-x_{n+2}+y_{n+1}=0$
- $x_{n+1}-2x_{n+2}+y_{n+2}=0$
- $2x_{n+1}-7x_{n+2}+y_{n+3}=0$
- $7x_{n+1}-x_{n+3}+4y_{n+1}=0$
- $x_{n+1}-x_{n+3}+2y_{n+2}=0$
- $x_{n+1}-7x_{n+3}+4y_{n+3}=0$
- $3x_{n+1}+2y_{n+1}-y_{n+2}=0$
- $12x_{n+1}+7y_{n+1}-y_{n+3}=0$
- $3x_{n+1}+7y_{n+2}-2y_{n+3}=0$
- $y_{n+1}+7x_{n+2}-2x_{n+3}=0$
- $y_{n+2}+2x_{n+2}-x_{n+3}=0$
- $y_{n+3}+x_{n+2}-2x_{n+3}=0$
- $y_{n+1}+3x_{n+2}-2y_{n+2}=0$
- $y_{n+1}+6x_{n+2}-y_{n+3}=0$
- $2y_{n+2}+3x_{n+2}-y_{n+3}=0$
- $7y_{n+2}-2y_{n+1}-3x_{n+3}=0$
- $7y_{n+3}-y_{n+1}-12x_{n+3}=0$
- $2y_{n+3}-y_{n+2}-3x_{n+3}=0$
- $y_{n+3}-4y_{n+2}+y_{n+1}=0$

2.4 Each of the following expressions represents a cubical integers

- $\frac{1}{(\alpha^2+6\alpha-3)}\left[[(2\alpha+6)x_{3n+4}-(24+4\alpha)x_{3n+3}]+3[(2\alpha+6)x_{n+2}-(24+4\alpha)x_{n+1}]\right]$
- $\frac{1}{2(\alpha^2+6\alpha-3)}\left[[(\alpha+3)x_{3n+5}-(45+7\alpha)x_{3n+3}]+3[(\alpha+3)x_{n+3}-(45+7\alpha)x_{n+1}]\right]$
- $\frac{1}{(\alpha^2+6\alpha-3)}\left[[(2\alpha+6)y_{3n+3}-12x_{3n+3}]+3[(2\alpha+6)y_{n+1}-12x_{n+1}]\right]$
- $\frac{1}{(\alpha^2+6\alpha-3)}\left[[(\alpha+3)y_{3n+4}-(21+3\alpha)x_{3n+3}]+3[(\alpha+3)y_{n+2}-(21+3\alpha)x_{n+1}]\right]$
- $\frac{1}{7(\alpha^2+6\alpha-3)}\left[2[(\alpha+3)y_{3n+5}-(78+12\alpha)x_{3n+3}]+6[(\alpha+3)y_{n+3}-(78+12\alpha)x_{n+1}]\right]$
- $\frac{1}{(\alpha^2+6\alpha-3)}\left[[(4\alpha+25)x_{3n+5}-(90+14\alpha)x_{3n+4}]+3[(4\alpha+24)x_{n+3}-(90+14\alpha)x_{n+2}]\right]$
- $\frac{1}{2(\alpha^2+6\alpha-3)}\left[[(4\alpha+24)y_{3n+3}-12x_{3n+4}]+3[(4\alpha+24)y_{n+1}-12x_{n+2}]\right]$
- $\frac{1}{(\alpha^2+6\alpha-3)}\left[[(4\alpha+24)y_{3n+4}-(42+6\alpha)x_{3n+4}]+3[(2\alpha+24)y_{n+2}-(42+6\alpha)x_{n+2}]\right]$
- $\frac{1}{2(\alpha^2+6\alpha-3)}\left[[(2\alpha+24)y_{3n+5}-(156+24\alpha)x_{3n+4}]+3[(4\alpha+24)y_{n+3}-(156+24\alpha)x_{n+2}]\right]$
- $\frac{1}{7(\alpha^2+6\alpha-3)}\left[[(14\alpha+90)y_{3n+3}-12x_{3n+5}]+3[(14\alpha+90)y_{n+1}-12x_{n+3}]\right]$
- $\frac{1}{2(\alpha^2+6\alpha-3)}\left[[(14\alpha+90)y_{3n+4}-(42+6\alpha)x_{3n+5}]+3[(14\alpha+90)y_{n+2}-(42+6\alpha)x_{n+3}]\right]$
- $\frac{1}{(\alpha^2+6\alpha-3)}\left[[(14\alpha+90)y_{3n+5}-(156+24\alpha)y_{3n+5}]+3[(14\alpha+90)y_{n+3}-(156+24\alpha)x_{n+3}]\right]$
- $\frac{1}{(\alpha^2+6\alpha-3)}\left[[(2\alpha+14)y_{3n+3}-4y_{3n+4}]+3[(2\alpha+14)y_{n+1}-4y_{n+2}]\right]$
- $\frac{1}{4(\alpha^2+6\alpha-3)}\left[[(8\alpha+52)y_{3n+3}-4y_{3n+5}]+3[(8\alpha+52)y_{n+1}-4y_{n+3}]\right]$
- $\frac{1}{(\alpha^2+6\alpha-3)}\left[[(8\alpha+52)y_{3n+4}-(14+2\alpha)y_{3n+5}]+3[(8\alpha+52)y_{n+2}-(14+2\alpha)y_{n+3}]\right]$

2.5 Each of the following expressions represents a Bi-quadratic Integer

- $\frac{1}{(\alpha^2+6\alpha-3)}\left[(2\alpha+6)x_{4n+5}-(24+4\alpha)x_{4n+4}+4[(2\alpha+6)x_{2n+3}-(24+4\alpha)x_{2n+2}]\right]+6$
- $\frac{1}{2(\alpha^2+6\alpha-3)}\left[(\alpha+3)x_{4n+6}-(45+7\alpha)x_{4n+4}+4[(\alpha+3)x_{2n+4}-(45+7\alpha)x_{2n+2}]\right]+6$
- $\frac{1}{(\alpha^2+6\alpha-3)}\left[(2\alpha+6)y_{4n+4}-12x_{4n+4}+4[(2\alpha+6)y_{2n+2}-12x_{2n+2}]\right]+6$
- $\frac{1}{(\alpha^2+6\alpha-3)}\left[(\alpha+3)y_{4n+5}-(21+3\alpha)x_{4n+4}+4[(\alpha+3)y_{2n+3}-(21+3\alpha)x_{2n+2}]\right]+6$
- $\frac{1}{7(\alpha^2+6\alpha-3)}\left[(2\alpha+6)y_{4n+6}-(156+24\alpha)x_{4n+4}+4[(2\alpha+6)y_{2n+4}-(156+12\alpha)x_{2n+2}]\right]+6$
- $\frac{1}{(\alpha^2+6\alpha-3)}\left[(4\alpha+24)x_{4n+6}-(90+14\alpha)x_{4n+5}+4[(8\alpha+52)x_{2n+4}-(14+2\alpha)x_{2n+3}]\right]+6$
- $\frac{1}{(\alpha^2+6\alpha-3)}\left[(4\alpha+24)x_{4n+4}-12x_{4n+5}+4[(4\alpha+24)y_{2n+2}-12x_{2n+3}]\right]+6$
- $\frac{1}{(\alpha^2+6\alpha-3)}\left[(4\alpha+24)y_{4n+5}-(42+6\alpha)x_{4n+5}+4[(4\alpha+24)y_{2n+3}-(42+6\alpha)x_{2n+3}]\right]+6$
- $\frac{1}{2(\alpha^2+6\alpha-3)}\left[(4\alpha+24)y_{4n+6}-(156+24\alpha)x_{4n+5}+4[(4\alpha+24)y_{2n+4}-(156+24\alpha)x_{2n+3}]\right]+6$
- $\frac{1}{7(\alpha^2+6\alpha-3)}\left[(14\alpha+90)y_{4n+4}-12x_{4n+6}+4[(14\alpha+90)y_{2n+2}-12x_{2n+4}]\right]+6$

➢ $\frac{1}{2(\alpha^2+6\alpha-3)}\left[(14\alpha+90)y_{4n+5}-(42+6\alpha)x_{4n+6}+4[(14\alpha+90)y_{2n+3}-(42+6\alpha)x_{2n+4}]\right]+6$

➢ $\frac{1}{(\alpha^2+6\alpha-3)}\left[(14\alpha+90)y_{4n+6}-(156+24\alpha)x_{4n+6}+4[(14\alpha+90)y_{2n+4}-(156+24\alpha)x_{2n+4}]\right]+6$

➢ $\frac{1}{(\alpha^2+6\alpha-3)}\left[(2\alpha+14)y_{4n+4}-4y_{4n+5}+4[(2\alpha+14)y_{2n+2}-4y_{2n+3}]\right]+6$

➢ $\frac{1}{4(\alpha^2+6\alpha-3)}\left[(8\alpha+52)y_{4n+4}-4y_{4n+6}+4[(8\alpha+52)y_{2n+2}-4y_{2n+4}]\right]+6$

➢ $\frac{1}{(\alpha^2+6\alpha-3)}\left[(8\alpha+52)y_{4n+5}-(14+2\alpha)y_{4n+6}+4[(8\alpha+52)y_{2n+3}-(14+2\alpha)y_{2n+4}]\right]+6$

2.6 Each of the following expressions represents a Quintic Integer

➢ $\frac{1}{(\alpha^2+6\alpha-3)}\left[(2\alpha+6)x_{5n+6}-(24+4\alpha)x_{5n+5}+5[(2\alpha+6)x_{3n+4}-(24+4\alpha)x_{3n+3}]+10[(2\alpha+6)x_{n+2}-(24+4\alpha)x_{n+1}]\right]$

➢ $\frac{1}{2(\alpha^2+6\alpha-3)}\left[(\alpha+3)x_{5n+7}-(45+7\alpha)x_{5n+5}+5[(\alpha+3)x_{3n+5}-(45+7\alpha)x_{3n+3}]+10[(\alpha+3)x_{n+3}-(45+7\alpha)x_{n+1}]\right]$

➢ $\frac{1}{(\alpha^2+6\alpha-3)}\left[(2\alpha+6)y_{5n+5}-12x_{5n+5}+5[(2\alpha+6)y_{3n+3}-12x_{3n+3}]+10[(2\alpha+6)y_{n+1}-12x_{n+1}]\right]$

➢ $\frac{1}{(\alpha^2+6\alpha-3)}\left[(\alpha+3)y_{5n+6}-(21+3\alpha)x_{5n+5}+5[(\alpha+3)y_{3n+4}-(21+3\alpha)x_{3n+3}]+10[(\alpha+3)y_{n+2}-(21+3\alpha)x_{n+1}]\right]$

- $\frac{1}{7(\alpha^2+6\alpha-3)}\Big[(2\alpha+6)y_{5n+7}-(156+24\alpha)x_{5n+5}+5[(2\alpha+6)y_{3n+5}-(156+24\alpha)x_{3n+3}]+10[(2\alpha+6)y_{n+3}-(156+24\alpha)x_{n+1}]\Big]$
- $\frac{1}{(\alpha^2+6\alpha-3)}\Big[(4\alpha+24)x_{5n+7}-(90+14\alpha)x_{5n+6}+5[(4\alpha+24)x_{3n+5}-(90+14\alpha)x_{3n+4}]+10[(4\alpha+24)x_{n+3}-(90+14\alpha)x_{n+2}]\Big]$
- $\frac{1}{2(\alpha^2+6\alpha-3)}\Big[(4\alpha+24)y_{5n+5}-12x_{5n+6}+5[(4\alpha+24)y_{3n+3}-12x_{3n+4}]+10[(4\alpha+24)y_{n+1}-12x_{n+2}]\Big]$
- $\frac{1}{(\alpha^2+6\alpha-3)}\Big[(4\alpha+24)y_{5n+6}-(42+6\alpha)x_{5n+6}+5[(4\alpha+24)y_{3n+4}-(42+6\alpha)x_{3n+4}]+10[(4\alpha+24)y_{n+2}-(42+6\alpha)x_{n+2}]\Big]$
- $\frac{1}{(\alpha^2+6\alpha-3)}\Big[(4\alpha+24)y_{5n+7}-(156+24\alpha)x_{5n+6}+5[(4\alpha+24)y_{3n+5}-(156+24\alpha)x_{3n+4}]+10[(4\alpha+24)y_{n+3}-(156+24\alpha)x_{n+2}]\Big]$
- $\frac{1}{7(\alpha^2+6\alpha-3)}\Big[(14\alpha+90)y_{5n+5}-12x_{5n+7}+5[(14\alpha+90)y_{3n+3}-12x_{3n+5}]+10[(14\alpha+90)y_{n+1}-12x_{n+3}]\Big]$
- $\frac{1}{2(\alpha^2+6\alpha-3)}\Big[(14\alpha+90)y_{5n+6}-(42+6\alpha)x_{5n+7}+5[(14\alpha+90)y_{3n+4}-(142+6\alpha)x_{3n+5}]+10[(14\alpha+90)y_{n+2}-(42+6\alpha)x_{n+3}]\Big]$
- $\frac{1}{(\alpha^2+6\alpha-3)}\Big[(14\alpha+90)y_{5n+7}-(156+24\alpha)x_{5n+7}+5[(14\alpha+90)y_{3n+5}-(156+24\alpha)x_{3n+5}]+10[(14\alpha+90)y_{n+3}-(156+24\alpha)x_{n+3}]\Big]$
- $\frac{1}{(\alpha^2+6\alpha-3)}\Big[(2\alpha+14)y_{5n+5}-4y_{5n+6}+5[(2\alpha+14)y_{3n+3}-4y_{3n+4}]+10[(2\alpha+14)y_{n+1}-4y_{n+2}]\Big]$
- $\frac{1}{4(\alpha^2+6\alpha-3)}\Big[(8\alpha+52)y_{5n+5}-4y_{5n+7}+5[(8\alpha+52)y_{3n+3}-4y_{3n+5}]+10[(8\alpha+52)y_{n+1}-4y_{n+3}]\Big]$

➢ $\frac{1}{(\alpha^2+6\alpha-3)}\left[(8\alpha+52)y_{5n+6}-(14+2\alpha)y_{5n+7}+5[(8\alpha+52)y_{3n+4}-(14+2\alpha)y_{3n+5}]+10[(8\alpha+52)y_{n+2}-(14+2\alpha)y_{n+3}]\right]$

2.7 Remarkable Observations:

1. Employing linear combinations among the solutions of (2.1), one may generate integer solutions for other choices of hyperbola which are presented in table :2.2 below

Table: 2.2 Hyperbola

S.no	Hyperbola	(P,Q)
1.	$P^2-Q^2=4(\alpha^2+6\alpha-3)^2$	$P=(2\alpha+6)x_{n+2}-(24+4\alpha)x_{n+1}$ $Q=(14+2\alpha)\sqrt{3}x_{n+1}-4\sqrt{3}x_{n+2}$
2.	$P^2-4Q^2=16(\alpha^2+6\alpha-3)^2$	$P=(\alpha+3)x_{n+3}-(45+7\alpha)x_{n+1}$ $Q=(13+2\alpha)\sqrt{3}x_{n+1}-\sqrt{3}x_{n+3}$
3.	$P^2-Q^2=4(\alpha^2+6\alpha-3)^2$	$P=(2\alpha+6)y_{n+1}-12x_{n+1}$ $Q=(6+2\alpha)\sqrt{3}x_{n+1}-4\sqrt{3}y_{n+1}$
4.	$P^2-Q^2=4(\alpha^2+6\alpha-3)^2$	$P=(\alpha+3)y_{n+2}-(21+3\alpha)x_{n+1}$ $Q=(12+2\alpha)\sqrt{3}x_{n+1}-2\sqrt{3}y_{n+2}$
5.	$P^2-Q^2=196(\alpha^2+6\alpha-3)^2$	$P=(2\alpha+6)y_{n+3}-(156+24\alpha)x_{n+1}$ $Q=(45+7\alpha)2\sqrt{3}x_{n+1}-4\sqrt{3}x_{n+3}$
6.	$3P^2-Q^2=12(\alpha^2+6\alpha-3)^2$	$P=(4\alpha+24)x_{n+3}-(90+14\alpha)x_{n+2}$ $Q=(52+8\alpha)\sqrt{3}x_{n+2}-(2\alpha+14)\sqrt{3}x_{n+3}$
7.	$P^2-Q^2=16(\alpha^2+6\alpha-3)^2$	$P=(4\alpha+24)y_{n+1}-12x_{n+2}$ $Q=(6+2\alpha)\sqrt{3}x_{n+2}-(2\alpha+14)\sqrt{3}y_{n+1}$
8.	$P^2-Q^2=4(\alpha^2+6\alpha-3)^2$	$P=(4\alpha+24)y_{n+2}-(42+6\alpha)x_{n+2}$ $Q=(24+4\alpha)\sqrt{3}x_{n+2}-(14+2\alpha)\sqrt{3}y_{n+2}$

9.	$P^2 - Q^2 = 16(\alpha^2 + 6\alpha - 3)^2$	$P = (4\alpha + 24)y_{n+3} - (156 + 24\alpha)x_{n+2}$ $Q = (90 + 14\alpha)\sqrt{3}x_{n+2} - (2\alpha + 14)\sqrt{3}y_{n+3}$
10.	$P^2 - Q^2 = 196(\alpha^2 + 6\alpha - 3)^2$	$P = (14\alpha + 90)y_{n+1} - 12x_{n+3}$ $Q = (2\alpha + 6)\sqrt{3}x_{n+3} - (8\alpha + 52)\sqrt{3}y_{n+1}$
11.	$P^2 - Q^2 = 16(\alpha^2 + 6\alpha - 3)^2$	$P = (14\alpha + 90)y_{n+2} - (42 + 6\alpha)x_{n+3}$ $Q = (24 + 4\alpha)\sqrt{3}x_{n+3} - (8\alpha + 52)\sqrt{3}y_{n+2}$
12.	$P^2 - Q^2 = 4(\alpha^2 + 6\alpha - 3)^2$	$P = (14\alpha + 90)y_{n+3} - (156 + 24\alpha)x_{n+3}$ $Q = (90 + 14\alpha)\sqrt{3}x_{n+3} - (8\alpha + 52)\sqrt{3}y_{n+3}$
13.	$3P^2 - Q^2 = 12(\alpha^2 + 6\alpha - 3)^2$	$P = (2\alpha + 14)y_{n+1} - 4y_{n+2}$ $Q = (6 + 2\alpha)y_{n+2} - (24 + 4\alpha)y_{n+1}$
14.	$3P^2 - Q^2 = 192(\alpha^2 + 6\alpha - 3)^2$	$P = (8\alpha + 52)y_{n+1} - 4y_{n+3}$ $Q = (6 + 2\alpha)y_{n+3} - (14\alpha + 90)y_{n+1}$
15.	$3P^2 - Q^2 = 12(\alpha^2 + 6\alpha - 3)^2$	$P = (8\alpha + 52)y_{n+2} - (14 + 2\alpha)y_{n+3}$ $Q = (24 + 4\alpha)y_{n+3} - (14\alpha + 90)y_{n+2}$

2. Employing linear combinations among the solutions of (2.1), one may generate integer solutions for other choices of parabola which are presented in the Table: 2.3 below:

Table: 2.3 Parabola

S. No	**Parabola**	**(R,Q)**
1	$R(\alpha^2 + 6\alpha - 3) - Q^2 = 4(\alpha^2 + 6\alpha - 3)^2$	$R = (2\alpha + 6)x_{n+3} - (24 + 4\alpha)x_{2n+2} + 2(\alpha^2 + 6\alpha - 3)$ $Q = (14 + 2\alpha)\sqrt{3}x_{n+1} - 4\sqrt{3}x_{n+2}$

2	$R(\alpha^2+6\alpha-3)-2Q^2 = 8(\alpha^2+6\alpha-3)^2$	$R=(\alpha+3)x_{2n+4}-(45+7\alpha)x_{2n+2}+4(\alpha^2+6\alpha-3)$ $Q=(13+2\alpha)\sqrt{3}x_{n+1}-\sqrt{3}x_{n+3}$
3	$R(\alpha^2+6\alpha-3)-Q^2 = 4(\alpha^2+6\alpha-3)^2$	$R=(2\alpha+6)Y_{2n+3}-12x_{2n+2}+2(\alpha^2+6\alpha-3)$ $Q=(6+2\alpha)\sqrt{3}x_{n+1}-4\sqrt{3}y_{n+1}$
4	$R(\alpha^2+6\alpha-3)-Q^2 = 4(\alpha^2+6\alpha-3)^2$	$R=(\alpha+3)Y_{2n+3}-(21+3\alpha)x_{2n+2}+2(\alpha^2+6\alpha-3)$ $Q=(12+2\alpha)\sqrt{3}x_{n+1}-2\sqrt{3}y_{n+2}$
5	$7R(\alpha^2+6\alpha-3)-Q^2 = 196(\alpha^2+6\alpha-3)^2$	$R=(2\alpha+6)Y_{2n+4}-(156+24\alpha)x_{2n+2}+14(\alpha^2+6\alpha-3)$ $Q=(45+7\alpha)2\sqrt{3}x_{n+1}-4\sqrt{3}x_{n+3}$
6	$R(\alpha^2+6\alpha-3)-Q^2 = 4(\alpha^2+6\alpha-3)^2$	$R=(24+4\alpha)x_{2n+4}-(14\alpha+90)x_{2n+3}+2(\alpha^2+6\alpha-3)$ $Q=(52+8\alpha)\sqrt{3}x_{n+2}-(2\alpha+14)\sqrt{3}x_{n+3}$
7	$2R(\alpha^2+6\alpha-3)-Q^2 = 16(\alpha^2+6\alpha-3)^2$	$R=[(24+4\alpha)y_{2n+2}-12(x_{2n+3})]+2(\alpha^2+6\alpha-3)$ $Q=(6+2\alpha)x_{n+2}-(2\alpha+14)\sqrt{3}y_{n+1}$
8	$R(\alpha^2+6\alpha-3)-Q^2 = 4(\alpha^2+6\alpha-3)^2$	$R=[(24+4\alpha)y_{2n+3}-(6\alpha+42)x_{2n+3}]+2(\alpha^2+6\alpha-3)$ $Q=(24+4\alpha)\sqrt{3}x_{n+2}-(14+2\alpha)\sqrt{3}y_{n+2}$
9	$2R(\alpha^2+6\alpha-3)-Q^2 = 16(\alpha^2+6\alpha-3)^2$	$R=(24+4\alpha)y_{2n+4}-(156+24\alpha)x_{2n+3}+4(\alpha^2+6\alpha-3)$ $Q=(90+14\alpha)\sqrt{3}x_{n+2}-(2\alpha+14)\sqrt{3}y_{n+3}$
10	$7R(\alpha^2+6\alpha-3)-Q^2 = 196(\alpha^2+6\alpha-3)^2$	$R=(90+14\alpha)y_{2n+2}-12x_{2n+4}+2(\alpha^2+6\alpha-3)$ $Q=(2\alpha+6)\sqrt{3}x_{n+3}-(8\alpha+52)\sqrt{3}y_{n+1}$
11	$2R(\alpha^2+6\alpha-3)-Q^2 = 16(\alpha^2+6\alpha-3)^2$	$R=(90+14\alpha)y_{2n+3}-(42+6\alpha)x_{2n+4}+4(\alpha^2+6\alpha-3)$

		$Q = (24+4\alpha)\sqrt{3}x_{n+3} - (8\alpha+52)\sqrt{3}y_{n+2}$
12	$R(\alpha^2+6\alpha-3) - Q^2 = 4(\alpha^2+6\alpha-3)^2$	$R = (90+14\alpha)y_{2n+4} - (156+24\alpha)x_{2n+4} + 2(\alpha^2+6\alpha-3)$ $Q = (90+14\alpha)\sqrt{3}x_{n+3} - (52+8\alpha)\sqrt{3}y_{n+3}$
13	$3R(\alpha^2+6\alpha-3) - Q^2 = 12(\alpha^2+6\alpha-3)^2$	$R = (2\alpha+14)y_{2n+2} - 4y_{2n+3} + 2(\alpha^2+6\alpha-3)$ $Q = (6+2\alpha)y_{n+2} - (24+4\alpha)y_{n+1}$
14	$12R(\alpha^2+6\alpha-3) - Q^2 = 192(\alpha^2+6\alpha-3)^2$	$R = (52+8\alpha)y_{2n+2} - 4y_{2n+4} + 8(\alpha^2+6\alpha-3)$ $Q = (6+2\alpha)y_{n+3} - (14\alpha+90)y_{n+1}$
15	$3R(\alpha^2+6\alpha-3) - Q^2 = 12(\alpha^2+6\alpha-3)^2$	$R = (52+8\alpha)y_{2n+3} - (2\alpha+14)y_{2n+4} + 2(\alpha^2+6\alpha-3)$ $Q = (24+4\alpha)y_{n+3} - (14\alpha+90)y_{n+2}$

2.8 Conclusion:

In this paper, we have presented infinitely many integer solutions for the Diophantine equations represented by the positive pell equations$y^2 = 3x^2 + \alpha^2 + 6\alpha - 3$. As the binary quadratic Diophantine equations are rich in variety, one may search for the other choices of pell equations and determine the solutions with the suitable properties.

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Chapter 3

A Search for Integer Solutions To Non-homogeneous Quadratic Equation With Two Unknowns $y^2 = 3x^2 + \alpha^2 + 10\alpha - 2$

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Abstract: A binary quadratic equation of the form $y^2 = Dx^2 + 1$, where D is non-square positive integer has been study by various mathematicians for it non-trivial integral solutions when D takes different integral values. The binary quadratic Diophantine equation represented by the positive pellian$y2 = 3x^2 + \alpha^2 + 10\alpha - 2$ is analysed for its non-zero distinct solutions. A few interesting relations among the solutions are given. Further,employing the solutions of the above hyperbola, the solutions of other choices of hyperbolas and parabolas are obtained.

Keywords: Binary quadratic, Hyperbola, Parabola, Pell equation, Integral solutions.
2010 Mathematics subject classification: 11D09

3.1 Introduction

A binary quadratic equation of the form $y^2 = Dx^2 + 1$, where D is non-square positive integer has been study by various mathematicians for it non-trivial integral solutions when D takes different integral values (Carmichael., 1959; Dickson., 1952; Mordell., 1969). For an extensive review of various problems, one may refer (Gopalan et.al., 2015; Mahalakshmi, Shanthi .,2023; Shanthi, Mahalakshmi .,2023; Shanthi, Gopalan.,2024). In this communication, yet another interesting hyperbola given by $y^2 = 3x^2 + \alpha^2 + 10\alpha - 2$ is considered and infinitely many integer solutions are obtained. A few interesting properties among the solutions are obtained. Further, employing the solutions of the above hyperbola, we have obtained solutions of other choices of hyperbola and parabola.

3.1 Method of analysis:

The Positive Pell equation representing hyperbola under consideration is

$$y^2 = 3x^2 + \alpha^2 + 10\alpha - 2 \qquad (3.1)$$

whose smallest positive integer solution is

$$x_0 = 3, y_0 = \alpha + 5$$

To obtain the other solutions of (3.1), consider the Pell equation

$$y^2 = 3x^2 + 1$$

whose general solution is given by

$$\tilde{x}_n = \frac{1}{2\sqrt{3}} g_n; \tilde{y}_n = \frac{1}{2} f_n$$

where

$$f_n = (2 + \sqrt{3})^{n+1} + (2 - \sqrt{3})^{n+1}$$

$$g_n = (2 + \sqrt{3})^{n+1} - (2 - \sqrt{3})^{n+1}$$

Applying Brahmagupta lemma between (x_0, y_0)&$(\tilde{x}_n, \tilde{y}_n)$the other integer solution of (3.1) are given by

$$x_{n+1} = \frac{9f_n}{6} + \frac{(\alpha + 5)\sqrt{3}g_n}{6}$$

$$y_{n+1} = \frac{(\alpha + 5)f_n}{2} + \frac{3\sqrt{3}g_n}{2}$$

The recurrence relations satisfied by x and y are given by

$$x_{n+1} - 4x_{n+2} + x_{n+3} = 0$$

$$y_{n+1} - 4y_{n+2} + y_{n+3} = 0$$

A few numerical examples are given in the following table: 3.1

Table: 3.1 Numerical values

n	x_n	y_n
0	3	$\alpha + 5$
1	$\alpha + 11$	$2\alpha + 19$
2	$4\alpha + 41$	$7\alpha + 71$
3	$15\alpha + 153$	$26\alpha + 265$
4	$56\alpha + 571$	$97\alpha + 989$

From the above table we observe some interesting properties among the solutions which are presented below:

3.3 Relations between solutions

- $x_{n+1} - 4x_{n+2} + x_{n+3} = 0$
- $2x_{n+1} - x_{n+2} + y_{n+1} = 0$
- $x_{n+1} - 2x_{n+2} + y_{n+2} = 0$
- $2x_{n+1} - 7x_{n+2} + y_{n+3} = 0$
- $7x_{n+1} - x_{n+3} + 4y_{n+1} = 0$
- $x_{n+1} - x_{n+3} + 2y_{n+2} = 0$
- $x_{n+1} - 7x_{n+3} + 4y_{n+3} = 0$
- $3x_{n+1} + 2y_{n+1} - y_{n+2} = 0$
- $12x_{n+1} + 7y_{n+1} - y_{n+3} = 0$
- $3x_{n+1} + 7y_{n+2} - 2y_{n+3} = 0$
- $y_{n+1} + 7x_{n+2} - 2x_{n+3} = 0$
- $y_{n+2} + 2x_{n+2} - x_{n+3} = 0$
- $y_{n+3} + x_{n+2} - 2x_{n+3} = 0$
- $y_{n+1} + 3x_{n+2} - 2y_{n+2} = 0$
- $y_{n+1} + 6x_{n+2} - y_{n+3} = 0$
- $2y_{n+2} + 3x_{n+2} - y_{n+3} = 0$
- $7y_{n+2} - 2y_{n+1} - 3x_{n+3} = 0$
- $7y_{n+3} - y_{n+1} - 12x_{n+3} = 0$
- $2y_{n+3} - y_{n+2} - 3x_{n+3} = 0$
- $y_{n+3} - 4y_{n+2} + y_{n+1} = 0$

3.4 Each of the following expressions represents a cubical integers

- $\frac{1}{(\alpha^2+10\alpha-2)}\left[[(2\alpha+10)x_{3n+4}-(38+4\alpha)x_{3n+3}]+3[(2\alpha+10)x_{n+2}-(38+4\alpha)x_{n+1}]\right]$
- $\frac{1}{2(\alpha^2+10\alpha-2)}\left[[(\alpha+5)x_{3n+5}-(71+7\alpha)x_{3n+3}]+3[(\alpha+5)x_{n+3}-(71+7\alpha)x_{n+1}]\right]$
- $\frac{1}{(\alpha^2+10\alpha-2)}\left[[(2\alpha+10)y_{3n+3}-18x_{3n+3}]+3[(2\alpha+10)y_{n+1}-18x_{n+1}]\right]$
- $\frac{1}{(\alpha^2+10\alpha-2)}\left[[(\alpha+5)y_{3n+4}-(33+3\alpha)x_{3n+3}]+3[(\alpha+5)y_{n+2}-(33+3\alpha)x_{n+1}]\right]$
- $\frac{1}{7(\alpha^2+10\alpha-2)}\left[[(2\alpha+10)y_{3n+5}-(246+24\alpha)x_{3n+3}]+3[(2\alpha+10)y_{n+3}-(246+24\alpha)x_{n+1}]\right]$
- $\frac{1}{3(\alpha^2+10\alpha-2)}\left[[(12\alpha+114)x_{3n+5}-(426+42\alpha)x_{3n+5}]+3[(12\alpha+114)x_{n+3}-(426+42\alpha)x_{n+2}]\right]$
- $\frac{1}{2(\alpha^2+10\alpha-2)}\left[[(4\alpha+38)y_{3n+3}-(18)x_{3n+4}]+3[(4\alpha+38)y_{n+1}-(18)x_{n+2}]\right]$
- $\frac{1}{(\alpha^2+10\alpha-2)}\left[[(4\alpha+38)y_{3n+4}-(6\alpha+66)x_{3n+4}]+3[(4\alpha+38)y_{n+2}-(6\alpha+66)x_{n+2}]\right]$
- $\frac{1}{2(\alpha^2+10\alpha-2)}\left[[(4\alpha+38)y_{3n+5}-(246+24\alpha)x_{3n+4}]+3[(4\alpha+38)y_{n+3}-(246+24\alpha)x_{n+2}]\right]$
- $\frac{1}{7(\alpha^2+10\alpha-2)}\left[[(14\alpha+142)y_{3n+3}-(18)x_{3n+5}]+3[(14\alpha+142)y_{n+1}-(18)x_{n+3}]\right]$
- $\frac{1}{2(\alpha^2+10\alpha-2)}\left[[(14\alpha+142)y_{3n+4}-(6\alpha+66)x_{3n+5}]+3[(14\alpha+142)y_{n+2}-(6\alpha+66)x_{n+3}]\right]$
- $\frac{1}{(\alpha^2+10\alpha-2)}\left[[(14\alpha+142)y_{3n+5}-(246+24\alpha)x_{3n+5}]+3[(14\alpha+142)y_{n+3}-(246+24\alpha)x_{n+3}]\right]$
- $\frac{1}{(\alpha^2+10\alpha-2)}\left[[(2\alpha+22)y_{3n+3}-6y_{3n+4}]+3[(2\alpha+22)y_{n+1}-6y_{n+2}]\right]$
- $\frac{1}{4(\alpha^2+10\alpha-2)}\left[[(8\alpha+82)y_{3n+3}-6y_{3n+5}]+3[(8\alpha+82)y_{n+1}-6y_{n+3}]\right]$
- $\frac{1}{(\alpha^2+10\alpha-2)}\left[[(8\alpha+82)y_{3n+4}-(22+2\alpha)y_{3n+5}]+3[(8\alpha+82)y_{n+2}-(22+2\alpha)y_{n+3}]\right]$

3.5 Each of the following expressions represents a Bi-quadratic Integer

- $\frac{1}{(\alpha^2+10\alpha-2)}\left[(2\alpha+10)x_{4n+5}-(38+4\alpha)x_{4n+4}+4[(2\alpha+10)x_{2n+3}-(38+4\alpha)x_{2n+2}]\right]+6$
- $\frac{1}{2(\alpha^2+10\alpha-2)}\left[(\alpha+5)x_{4n+6}-(71+7\alpha)x_{4n+4}+4[(\alpha+5)x_{2n+4}-(71+7\alpha)x_{2n+2}]\right]+6$
- $\frac{1}{(\alpha^2+10\alpha-2)}\left[(2\alpha+10)y_{4n+4}-18x_{4n+4}+4[(2\alpha+10)y_{2n+2}-18x_{2n+2}]\right]+6$
- $\frac{1}{(\alpha^2+10\alpha-2)}\left[(\alpha+5)y_{4n+5}-(33+3\alpha)x_{4n+4}+4[(\alpha+5)y_{2n+3}-(33+3\alpha)x_{2n+2}]\right]+6$
- $\frac{1}{7(\alpha^2+10\alpha-2)}\left[(2\alpha+10)y_{4n+6}-(246+24\alpha)x_{4n+4}+4[(2\alpha+10)y_{2n+4}-(246+24\alpha)x_{2n+2}]\right]+6$
- $\frac{1}{3(\alpha^2+10\alpha-2)}\left[(12\alpha+114)x_{4n+6}-(426+42\alpha)x_{4n+5}+4[(12\alpha+114)x_{2n+4}-(426+42\alpha)x_{2n+3}]\right]+6$
- $\frac{1}{2(\alpha^2+10\alpha-2)}\left[(4\alpha+38)y_{4n+4}-18x_{4n+5}+4[(4\alpha+38)y_{2n+2}-18x_{2n+3}]\right]+6$
- $\frac{1}{(\alpha^2+10\alpha-2)}\left[(4\alpha+38)y_{4n+5}-(66+6\alpha)x_{4n+5}+4[(4\alpha+38)y_{2n+3}-(66+6\alpha)x_{2n+3}]\right]+6$
- $\frac{1}{2(\alpha^2+10\alpha-2)}\left[(4\alpha+38)y_{4n+6}-(246+24\alpha)x_{4n+5}+4[(4\alpha+38)y_{2n+4}-(246+24\alpha)x_{2n+3}]\right]+6$
- $\frac{1}{7(\alpha^2+10\alpha-2)}\left[(14\alpha+142)y_{4n+4}-18x_{4n+6}+4[(14\alpha+142)y_{2n+2}-18x_{2n+4}]\right]+6$
- $\frac{1}{2(\alpha^2+10\alpha-2)}\left[(14\alpha+142)y_{4n+5}-(66+6\alpha)x_{4n+6}+4[(14\alpha+142)y_{2n+3}-(66+6\alpha)x_{2n+4}]\right]+6$
- $\frac{1}{(\alpha^2+10\alpha-2)}\left[(14\alpha+142)y_{4n+6}-(246+24\alpha)x_{4n+6}+4[(14\alpha+142)y_{2n+4}-(2466+24\alpha)x_{2n+4}]\right]+6$
- $\frac{1}{(\alpha^2+10\alpha-2)}\left[(2\alpha+22)y_{4n+4}-6y_{4n+5}+4[(2\alpha+22)y_{2n+2}-6y_{2n+3}]\right]+6$
- $\frac{1}{4(\alpha^2+10\alpha-2)}\left[(8\alpha+82)y_{4n+4}-6y_{4n+6}+4[(8\alpha+82)y_{2n+2}-6y_{2n+4}]\right]+6$

- $\frac{1}{(\alpha^2+10\alpha-2)}[(8\alpha+82)y_{4n+5}-(22+2\alpha)y_{4n+6}+4[(8\alpha+82)y_{2n+3}-(22+2\alpha)y_{2n+4}]]+6$

3.6 Each of the following expressions represents a Quintic Integer

- $\frac{1}{(\alpha^2+10\alpha-2)}[(2\alpha+10)x_{5n+6}-(38+4\alpha)x_{5n+5}+5[(2\alpha+10)x_{5n+6}-(38+4\alpha)x_{5n+5}]+10[(2\alpha+2)x_{n+2}-(38+4\alpha)x_{n+1}]]$
- $\frac{1}{2(\alpha^2+10\alpha-2)}[(\alpha+5)x_{5n+7}-(71+7\alpha)x_{5n+5}+5[(\alpha+5)x_{3n+2}-(71+7\alpha)x_{3n+3}]+10[(\alpha+5)x_{n+3}-(71+7\alpha)x_{n+1}]]$
- $\frac{1}{(\alpha^2+10\alpha-2)}[(2\alpha+10)y_{5n+5}-18x_{5n+5}+5[(2\alpha+10)y_{3n+3}-18x_{3n+3}]+10[(2\alpha+10)y_{n+1}-18x_{n+1}]]$
- $\frac{1}{(\alpha^2+10\alpha-2)}[(\alpha+5)y_{5n+6}-(33+3\alpha)x_{5n+5}+5[(\alpha+5)y_{3n+4}-(33+3\alpha)x_{3n+3}]+10[(\alpha+5)y_{n+2}-(33+3\alpha)x_{n+1}]]$
- $\frac{1}{7(\alpha^2+10\alpha-2)}[(2\alpha+10)y_{5n+7}-(246+24\alpha)x_{5n+5}+5[(2\alpha+10)y_{3n+5}-(246+24\alpha)x_{3n+3}]+10[(2\alpha+10)y_{n+3}-(246+24\alpha)x_{n+1}]]$
- $\frac{1}{3(\alpha^2+10\alpha-2)}[(12\alpha+114)x_{5n+7}-(426+42\alpha)x_{5n+6}+5[(12\alpha+114)x_{3n+5}-(426+42\alpha)x_{3n+5}]+10[(12\alpha+114)x_{n+3}-(42+426\alpha)x_{n+2}]]$
- $\frac{1}{2(\alpha^2+10\alpha-2)}[(4\alpha+38)y_{5n+5}-18x_{5n+6}+5[(4\alpha+38)y_{3n+3}-18x_{3n+4}]+10[(4\alpha+38)y_{n+1}-18x_{n+2}]]$
- $\frac{1}{(\alpha^2+10\alpha-2)}[(4\alpha+38)y_{5n+6}-(66+6\alpha)x_{5n+6}+5[(4\alpha+38)y_{3n+4}-(66+6\alpha)x_{3n+4}]+10[(4\alpha+38)y_{n+2}-(66+6\alpha)x_{n+2}]]$
- $\frac{1}{2(\alpha^2+10\alpha-2)}[(4\alpha+38)y_{5n+7}-(246+24\alpha)x_{5n+6}+5[(4\alpha+38)y_{3n+5}-(246+24\alpha)x_{3n+4}]+10[(4\alpha+38)y_{n+3}-(246+24\alpha)x_{n+2}]]$
- $\frac{1}{7(\alpha^2+10\alpha-2)}[(14\alpha+142)y_{5n+5}-18x_{5n+7}+5[(14\alpha+142)y_{3n+3}-18x_{3n+5}]+10[(14\alpha+142)y_{n+1}-18x_{n+3}]]$

➢ $\frac{1}{2(\alpha^2+10\alpha-2)}\left[(14\alpha+142)y_{5n+6}-(66+6\alpha)x_{5n+7}+5[(14\alpha+142)y_{3n+4}-(66+6\alpha)x_{3n+5}]+10[(14\alpha+142)y_{n+2}-(66+6\alpha)x_{n+3}]\right]$

➢ $\frac{1}{(\alpha^2+10\alpha-2)}\left[(14\alpha+142)y_{5n+7}-(246+24\alpha)x_{5n+7}+5[(14\alpha+142)y_{3n+5}-(246+24\alpha)x_{3n+5}]+10[(14\alpha+142)y_{n+3}-(246+24\alpha)x_{n+3}]\right]$

➢ $\frac{1}{(\alpha^2+10\alpha-2)}\left[(2\alpha+22)y_{5n+5}-6y_{5n+6}+5[(2\alpha+22)y_{3n+3}-6y_{3n+4}]+10[(2\alpha+22)y_{n+1}-6y_{n+2}]\right]$

➢ $\frac{1}{4(\alpha^2+10\alpha-2)}\left[(8\alpha+82)y_{5n+5}-6y_{5n+7}+5[(8\alpha+82)y_{3n+3}-6y_{3n+5}]+10[(8\alpha+82)y_{n+1}-6y_{n+3}]\right]$

➢ $\frac{1}{(\alpha^2+10\alpha-2)}\left[(8\alpha+82)y_{5n+6}-(22+2\alpha)y_{5n+7}+5[(8\alpha+82)y_{3n+4}-(22+2\alpha)y_{3n+5}]+10[(8\alpha+82)y_{n+2}-(22+2\alpha)y_{n+3}]\right]$

3.7 Remarkable Observations:

1. Employing linear combinations among the solutions of (3.1), one may generate integer solutions for other choices of hyperbola which are presented in table :3.2 below

Table:3.2 Hyperbola

S.No	Hyperbola	(P,Q)
1.	$3P^2-Q^2=12(\alpha^2+10\alpha-2)^2$	$P=(2\alpha+10)x_{n+2}-(38+4\alpha)x_{n+1}$ $Q=(66+6\alpha)x_{n+1}-18x_{n+2}$
2.	$3P^2-Q^2=48(\alpha^2+10\alpha-2)^2$	$P=(\alpha+5)x_{n+3}-(71+7\alpha)x_{n+1}$ $Q=(123+12\alpha)x_{n+1}-9x_{n+3}$
3.	$3P^2-Q^2=12(\alpha^2+10\alpha-2)^2$	$P=(2\alpha+10)y_{n+1}-18x_{n+1}$ $Q=(30+6\alpha)x_{n+1}-18y_{n+1}$

4.	$3P^2 - Q^2 = 12(\alpha^2 + 10\alpha - 2)^2$	$P = (\alpha + 5)y_{n+2} - (33 + 3\alpha)x_{n+1}$ $Q = (57 + 6\alpha)x_{n+1} - 9y_{n+2}$
5.	$3P^2 - Q^2 = 588(\alpha^2 + 10\alpha - 2)^2$	$P = (2\alpha + 10)y_{n+3} - (246 + 24\alpha)x_{n+1}$ $Q = (426 + 42\alpha)x_{n+1} - 18y_{n+3}$
6.	$3P^2 - Q^2 = 108(\alpha^2 + 10\alpha - 2)^2$	$P = (12\alpha + 114)x_{n+3} - (426 + 42\alpha)x_{n+2}$ $Q = (738 + 72\alpha)x_{n+2} - (18\alpha + 198)x_{n+3}$
7.	$3P^2 - Q^2 = 48(\alpha^2 + 10\alpha - 2)^2$	$P = (4\alpha + 38)y_{n+1} - 18x_{n+2}$ $Q = (30 + 6\alpha)x_{n+2} - (6\alpha + 66)y_{n+1}$
8.	$3P^2 - Q^2 = 12(\alpha^2 + 10\alpha - 2)^2$	$P = (4\alpha + 38)y_{n+2} - (6\alpha + 66)x_{n+2}$ $Q = (114 + 12\alpha)x_{n+2} - (66 + 6\alpha)y_{n+2}$
9.	$3P^2 - Q^2 = 48(\alpha^2 + 10\alpha - 2)^2$	$P = (4\alpha + 38)y_{n+3} - (246 + 24\alpha)x_{n+2}$ $Q = (426 + 42\alpha)x_{n+2} - (6\alpha + 66)y_{n+3}$
10.	$3P^2 - Q^2 = 588(\alpha^2 + 10\alpha - 2)^2$	$P = (14\alpha + 142)y_{n+1} - 18x_{n+3}$ $Q = (6\alpha + 30)x_{n+3} - (24\alpha + 246)y_{n+1}$
11.	$3P^2 - Q^2 = 48(\alpha^2 + 10\alpha - 2)^2$	$P = (14\alpha + 142)y_{n+2} - (66 + 6\alpha)x_{n+3}$ $Q = (114 + 12\alpha)x_{n+3} - (24\alpha + 246)y_{n+2}$

12.	$3P^2 - Q^2 = 12(\alpha^2 + 10\alpha - 2)^2$	$P = (14\alpha + 142)y_{n+3} - (24\alpha + 246)x_{n+3}$ $Q = (426 + 42\alpha)x_{n+3} - (24\alpha + 66)y_{n+3}$
13.	$3P^2 - Q^2 = 12(\alpha^2 + 10\alpha - 2)^2$	$P = (2\alpha + 22)y_{n+1} - 6y_{n+2}$ $Q = (10 + 2\alpha)y_{n+2} - (38 + 4\alpha)y_{n+1}$
14.	$3P^2 - Q^2 = 192(\alpha^2 + 10\alpha - 2)^2$	$P = (8\alpha + 82)y_{n+1} - 6y_{n+3}$ $Q = (10 + 2\alpha)y_{n+3} - (14\alpha + 142)y_{n+1}$
15.	$3P^2 - Q^2 = 12(\alpha^2 + 10\alpha - 2)^2$	$P = (8\alpha + 82)y_{n+2} - (22 + 2\alpha)y_{n+3}$ $Q = (38 + 4\alpha)y_{n+3} - (14\alpha + 142)y_{n+2}$

2.Employing linear combinations among the solutions of (3.1), one may generate integer solutions for other choices of parabola which are presented in the Table: 3.3 below:

Table: 3.3 Parabola

s.no	Parabola	(R,Q)
1	$3R(\alpha^2 + 10\alpha - 2) - Q^2 = 12(\alpha^2 + 10\alpha - 2)^2$	$R = (2\alpha + 10)x_{2n+3} - (4\alpha + 38)x_{2n+2} + 2(\alpha^2 + 10\alpha - 2)$ $Q = (66 + 6\alpha)x_{n+1} - 18x_{n+2}$
2	$6R(\alpha^2 + 10\alpha - 2) - Q^2 = 48(\alpha^2 + 10\alpha - 2)^2$	$R = (\alpha + 5)x_{4n+1} - (7\alpha + 71)x_{2n+2} + 4(\alpha^2 + 10\alpha - 2)$ $Q = (123 + 12\alpha)x_{n+1} - 9x_{n+3}$
3	$3R(\alpha^2 + 10\alpha - 2) - Q^2 = 12(\alpha^2 + 10\alpha - 2)^2$	$R = (2\alpha + 10)y_{2n+2} - 18x_{2n+2} + 2(\alpha^2 + 10\alpha - 2)$ $Q = (30 + 6\alpha)x_{n+1} - 18y_{n+1}$

4	$3R(\alpha^2+10\alpha-2)-Q^2 = 12(\alpha^2+10\alpha-2)^2$	$R=(\alpha+5)y_{2n+3}-(3\alpha+33)x_{2n+2}+2(\alpha^2+10\alpha-2)$ $Q=(57+6\alpha)x_{n+1}-9y_{n+2}$
5	$21R(\alpha^2+10\alpha-2)-Q^2 = 588(\alpha^2+10\alpha-2)^2$	$R=(2\alpha+10)y_{n+3}-(24\alpha+246)x_{n+1}+2(\alpha^2+10\alpha-2)$ $Q=(426+42\alpha)x_{n+1}-18y_{n+2}$
6	$9R(\alpha^2+10\alpha-2)-Q^2 = 108(\alpha^2+10\alpha-2)^2$	$R=[(114+12\alpha)x_{2n+4}-(426+42\alpha)x_{2n+3}]+6(\alpha^2+10\alpha-2)$ $Q=6(738+72\alpha)x_{n+2}-(18\alpha+198)x_{n+3}$
7	$6R(\alpha^2+10\alpha-2)-Q^2 = 48(\alpha^2+10\alpha-2)^2$	$R=[(38+4\alpha)y_{2n+2}-18(x_{2n+3})]+2(\alpha^2+10\alpha-2)$ $Q=(36+6\alpha)x_{n+2}-(6\alpha+66)y_{n+1}$
8	$3R(\alpha^2+10\alpha-2)-Q^2 = 12(\alpha^2+10\alpha-2)^2$	$R=[(38+4\alpha)y_{2n+3}-(6\alpha+66)x_{2n+3}]+2(\alpha^2+10\alpha-2)$ $Q=(114+12\alpha)x_{n+2}-(66+6\alpha)y_{n+2}$
9	$6R(\alpha^2+10\alpha-2)-Q^2 = 48(\alpha^2+10\alpha-2)^2$	$R=(38+4\alpha)y_{2n+4}-(246+24\alpha)x_{2n+3}+2(\alpha^2+10\alpha-2)$ $Q=(426+42\alpha)x_{n+2}-(6\alpha+66)y_{n+3}$
10	$21R(\alpha^2+10\alpha-2)-Q^2 = 588(\alpha^2+10\alpha-2)^2$	$R=(142+14\alpha)y_{2n+2}-18x_{2n+4}+14(\alpha^2+10\alpha-2)$ $Q=(6\alpha+30)x_{n+3}-(24\alpha+246)y_{n+1}$
11	$6R(\alpha^2+10\alpha-2)-Q^2 = 48(\alpha^2+10\alpha-2)^2$	$R=(142+14\alpha)y_{2n+3}-(66+6\alpha)x_{2n+4}+4(\alpha^2+10\alpha-2)$ $Q=(114+12\alpha)x_{n+3}-(24\alpha+246)y_{n+2}$

12	$3R(\alpha^2+10\alpha-2)-Q^2 = 12(\alpha^2+10\alpha-2)^2$	$R=(142+14\alpha)y_{2n+4}-(246+24\alpha)x_{2n+4}+2(\alpha^2+10\alpha-2)$ $Q=(426+42\alpha)x_{n+3}-(246+24\alpha)y_{n+3}$
13	$3R(\alpha^2+10\alpha-2)-Q^2 = 12(\alpha^2+10\alpha-2)^2$	$R=(2\alpha+22)y_{2n+2}-6y_{2n+3}+2(\alpha^2+10\alpha-2)$ $Q=(10+2\alpha)y_{n+2}-(38+4\alpha)y_{n+1}$
14	$12R(\alpha^2+10\alpha-2)-Q^2 = 192(\alpha^2+10\alpha-2)^2$	$R=(82+8\alpha)y_{2n+2}-6y_{2n+4}+8(\alpha^2+10\alpha-2)$ $Q=(10+2\alpha)y_{n+3}-(14\alpha+142)y_{n+1}$
15	$3R(\alpha^2+10\alpha-2)-Q^2 = 12(\alpha^2+10\alpha-2)^2$	$R=(82+8\alpha)y_{2n+3}-(2\alpha+22)y_{2n+4}+2(\alpha^2+10\alpha-2)$ $Q=(38+4\alpha)y_{n+3}-(14\alpha+142)y_{n+2}$

3.8 Conclusion:

In this paper, we have presented infinitely many integer solutions for the Diophantine equationsrepresentedby the positive pell equations $y^2=3x^2+\alpha^2+10\alpha-2$.As the binary quadratic Diophantine equations are rich in variety, one may search for the other choices of pell equations and determine the solutions with the suitable properties.

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Chapter 4

Technique To Solve Pell-Like Equation $6x^2 - 5y^2 = 6$

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Abstract: The non-homogeneous second degree equation with two unknowns represented by the Pell- like equation $6x^2 - 5y^2 = 6$ is studied for finding its distinct integer solutions. A few interesting properties between the above solutions are presented.

Keywords: Binary quadratic, Hyperbola, Parabola, Pell equation, Integral solutions.
2010 Mathematics subject classification: 11D09

4.1 Introduction

The second degree equations of the form $ax^2 - by^2 = N, (a, b, Z \neq 0)$ are rich in variety (Carmichael., 1959; Dickson., 1952; Mordell., 1969).and have been analyzed by many mathematicians for their respective integer solutions for particular values of a, b and N. In this context, one may refer (Gopalan et.al., 2021; Mahalakshmi, Shalini .,2023; Shanthi, Parkavi .,2023 ;Shanthi,Indhumuki.,2023).

This communication concerns with the problem of obtaining non-zero distinct integer solutions to the second degree equation given by $6x^2 - 5y^2 = 6$ representing hyperbola. A few interesting relations among its solutions are presented. Knowing an integral solution of the given hyperbola, integer solutions for other choices of hyperbola and parabolas are presented.

4.2 Method of analysis:

The second degree equation under consideration is

$$6x^2 - 5y^2 = 6 \qquad (4.1)$$

It is to be noted that (4.1) represents a hyperbola

Taking $x = X + 5T,\ y = X + 6T$ (4.2)

In (4.1), it reduced to the equation

$$X^2 = 30T^2 + 6 \qquad (4.3)$$

The smallest positive integer solution (T_0, X_0) of (4.3) is

$$T_0 = 1\ ,\ X_0 = 6$$

To obtain the other solutions of (4.3), consider the pellian equation

$$X^2 = 30T^2 + 1 \qquad (4.4)$$

whose smallest positive integer solution is

$$\tilde{T}_0 = 2, \quad \tilde{X}_0 = 11$$

The general solution $(\tilde{T}_n, \tilde{X}_n)$ of (4.4) is given by

$$\tilde{X}_n + \sqrt{30}\tilde{T}_n = \left(11 + 2\sqrt{30}\right)^{n+1} \quad , n=0,1,2,3\ldots.. \qquad (4.5)$$

Since irrational roots occur in pairs, we have

$$\tilde{X}_n - \sqrt{30}\tilde{T}_n = \left(11 - 2\sqrt{30}\right)^{n+1} \quad , n=0,1,2,3\ldots\ldots \qquad (4.6)$$

From (4.5) and (4.6), solving for $\tilde{X}_n, \tilde{T}_n$, we have

$$\tilde{X}_n = \frac{1}{2}\left[\left(11 + 2\sqrt{30}\right)^{n+1} + \left(11 - 2\sqrt{30}\right)^{n+1}\right] = \frac{1}{2} f_n$$

$$\tilde{T}_n = \frac{1}{2\sqrt{30}}\left[\left(11 + 2\sqrt{30}\right)^{n+1} - \left(11 - 2\sqrt{30}\right)^{n+1}\right] = \frac{1}{2\sqrt{30}} g_n$$

Applying Brahmagupta lemma between the solutions $(T_0, X_0)$and $(\tilde{T}_n, \tilde{X}_n)$, the general solution (T_{n+1}, X_{n+1}) of (4.3) is found to be

$$T_{n+1} = X_0 \widetilde{T}_n + T_0 \widetilde{X}_n = \frac{\sqrt{30}}{10} g_n + \frac{1}{2} f_n \tag{4.7}$$

$$X_{n+1} = X_0 \widetilde{X}_n + 30 T_0 \widetilde{T}_n = 3 f_n + \frac{\sqrt{30}}{2} g_n \tag{4.8}$$

Using (4.7) and (4.8) in (4.2) we have,

$$2x_{n+1} = 11 f_n + 2\sqrt{30} g_n \tag{4.9}$$

$$10 y_{n+1} = 60 f_n + 11\sqrt{30} g_n \tag{4.10}$$

Thus (4.9) and (4.10) represent the integer solutions of the hyperbola (4.1).

A few numerical values are given in the following Table:4.1

Table :4.1 Numerical values

n	f_n	g_n	x_{n+1}	y_{n+1}
-1	2	0	11	12
0	22	$4\sqrt{30}$	241	264
1	482	$88\sqrt{30}$	5291	5796
2	10582	$1932\sqrt{30}$	116161	127248
3	232322	$42416\sqrt{30}$	2550251	2793660
4	5100502	$931220\sqrt{30}$	55989361	61333272

Recurrence relation for x and y are:

$$x_{n+3} - 22x_{n+2} + x_{n+1} = 0 \quad , \quad n = -1,0,1\ldots.$$

$$y_{n+3} - 22y_{n+2} + y_{n+1} = 0 \quad , \quad n = -1,0,1\ldots$$

4.3 A few interesting relations among the solutions are given below.

- $2x_{n+3} + 2x_{n+1} - 44x_{n+2} = 0$
- $10y_{n+1} - x_{n+2} + 11x_{n+1} = 0$
- $10y_{n+2} - 11x_{n+2} + x_{n+1} = 0$
- $10y_{n+3} - 241x_{n+2} + 11x_{n+1} = 0$
- $220y_{n+1} - x_{n+3} + 241x_{n+1} = 0$
- $20y_{n+2} - x_{n+3} + x_{n+1} = 0$
- $220y_{n+3} - 241x_{n+3} + x_{n+1} = 0$
- $10y_{n+2} - 120x_{n+1} - 110y_{n+1} = 0$
- $10y_{n+3} - 2640x_{n+1} - 2410y_{n+1} = 0$
- $110y_{n+3} - 120x_{n+1} - 2410y_{n+2} = 0$
- $10y_{n+1} - 11x_{n+3} + 241x_{n+2} = 0$
- $10y_{n+2} - x_{n+3} + 11x_{n+2} = 0$
- $10y_{n+3} - 11x_{n+3} + x_{n+2} = 0$
- $110y_{n+2} - 120x_{n+2} - 10y_{n+1} = 0$
- $110y_{n+3} - 2640x_{n+2} - 110y_{n+1} = 0$
- $10y_{n+3} - 120x_{n+2} - 110y_{n+2} = 0$
- $2410y_{n+2} - 120x_{n+3} - 110y_{n+1} = 0$
- $2410y_{n+3} - 2640x_{n+3} - 10y_{n+1} = 0$
- $110y_{n+3} - 120x_{n+3} - 10y_{n+2} = 0$
- $60y_{n+3} - 1320y_{n+2} + 60y_{n+1} = 0$

4.4 Each of following expressions represents a cubic integer

- $44x_{3n+3} - 2x_{3n+4} + 3(44x_{n+1} - 2x_{n+2})$
- $\frac{1}{11}[(483x_{3n+3} - x_{3n+5}) + 3(483x_{n+1} - x_{n+3})]$
- $22x_{3n+3} - 20y_{3n+3} + 3(22x_{n+1} - 20y_{n+1})$
- $\frac{1}{11}[(482x_{3n+3} - 20y_{3n+4}) + 3(482x_{n+1} - 20y_{n+2})]$
- $10582x_{3n+3} - 20y_{3n+5} + 3(10582x_{n+1} - 20y_{n+3})$
- $966x_{3n+4} - 44x_{3n+5} + 3(966x_{n+2} - 44x_{n+3})$
- $2x_{3n+4} - 40y_{3n+3} + 3(2x_{n+2} - 40y_{n+1})$
- $482x_{3n+4} - 440y_{3n+4} + 3(482x_{n+2} - 440y_{n+2})$

- $962x_{3n+4} - 40y_{3n+5} + 3(962x_{n+2} - 40y_{n+3})$
- $\frac{1}{241}[(22x_{3n+5} - 9660y_{3n+3}) + 3(22x_{n+3} - 9660y_{n+1})]$
- $\frac{1}{11}[(482x_{3n+5} - 9660y_{3n+4}) + 3(482x_{n+3} - 9660y_{n+2})]$
- $10582x_{3n+5} - 9660y_{3n+5} + 3(10582x_{n+3} - 9660y_{n+3})$
- $\frac{1}{6}[(11y_{3n+4} - 241y_{3n+3}) + 3(11y_{n+2} - 241y_{n+1})]$
- $\frac{1}{12}[(y_{3n+5} - 481y_{3n+3}) + 3(y_{n+3} - 481y_{n+1})]$
- $\frac{1}{6}[(241y_{3n+5} - 5291y_{3n+4}) + 3(241y_{n+3} - 5291y_{n+2})]$

4.5 Each of the following expressions represents a Biquadratic Integer

- $44x_{4n+4} - 2x_{4n+5} + 4(44x_{2n+2} - 2x_{2n+3}) + 6$
- $\frac{1}{11}[(483x_{4n+4} - x_{4n+6}) + 4(483x_{2n+2} - x_{2n+4})] + 66$
- $22x_{4n+4} - 20y_{4n+4} + 4(22x_{2n+2} - 20y_{2n+2}) + 6$
- $\frac{1}{11}[(482x_{4n+4} - 20y_{4n+5}) + 4(482x_{2n+2} - 20y_{2n+3}) + 66]$
- $\frac{1}{241}[(10582x_{4n+4} - 20y_{4n+6}) + 4(10582x_{2n+2} - 20y_{2n+4}) + 1446]$
- $966x_{4n+5} - 44x_{4n+6} + 4(966x_{2n+3} - 44x_{2n+4}) + 6$
- $2x_{4n+5} - 40y_{4n+4} + 4(2x_{2n+3} - 40y_{2n+2}) + 6$
- $482x_{4n+5} - 440y_{4n+5} + 4(482x_{2n+3} - 440y_{2n+3}) + 6$
- $962x_{4n+5} - 40y_{4n+6} + 4(962x_{2n+3} - 40y_{2n+4}) + 6$
- $\frac{1}{241}[(22x_{4n+6} - 9660y_{4n+4}) + 4(22x_{2n+4} - 9660y_{2n+2}) + 1446]$
- $\frac{1}{11}[(482x_{4n+6} - 9660y_{4n+5}) + 4(482x_{2n+4} - 9660y_{2n+3}) + 66]$
- $\frac{1}{6}[(11y_{4n+5} - 241y_{4n+4}) + 4(11y_{2n+3} - 241y_{2n+2}) + 36]$
- $\frac{1}{12}[(y_{4n+6} - 481y_{4n+4}) + 4(y_{2n+4} - 481y_{2n+2}) + 72]$
- $\frac{1}{6}[(241y_{4n+6} - 5291y_{4n+5}) + 4(241y_{2n+4} - 5291y_{2n+3}) + 36]$

4.6 Each of the following expressions represents a Quintic Integer

- $44x_{5n+5} - 2x_{5n+6} + 5[(44x_{3n+3} - 2x_{3n+4}) + 3(44x_{n+1} - 2x_{n+2})] - 5(44x_{n+1} - 2x_{n+2})$

- $\frac{1}{11}[(483x_{5n+5} - x_{5n+7}) + 5[(483x_{3n+3} - x_{3n+5}) + 3(483x_{n+1} - x_{n+3})] - 5(483x_{n+1} - x_{n+3})]$

- $22x_{5n+5} - 20y_{5n+5} + 5[(22x_{3n+3} - 20y_{3n+3}) + 3(22x_{n+1} - 20y_{n+1})] - 5(22x_{n+1} - 20y_{n+1})$

- $\frac{1}{11}[(482x_{5n+5} - 20y_{5n+6}) + 5[(482x_{3n+3} - 20y_{3n+4}) + 3(482x_{n+1} - 20y_{n+2})] - 5(482x_{n+1} - 20y_{n+2})]$

- $\frac{1}{241}[(10582x_{5n+5} - 20y_{5n+7}) + 5[(10582x_{3n+3} - 20y_{3n+5}) + 3(10582x_{n+1} - 20y_{n+3})] - 5(10582x_{n+1} - 20y_{n+3})]$

- $966x_{5n+6} - 44x_{5n+7} + 5[(966x_{3n+4} - 44x_{3n+5}) + 3(966x_{n+2} - 44x_{n+3})] - 5(966x_{n+2} - 44x_{n+3})$
-
- $2x_{5n+6} - 40y_{5n+5} + 5[(2x_{3n+4} - 40y_{3n+3}) + 3(2x_{n+2} - 40y_{n+1})] - 2(2x_{n+2} - 40y_{n+1})$

- $482x_{5n+6} - 440y_{5n+6} + 5[(482x_{3n+4} - 440y_{3n+4}) + 3(482x_{n+2} - 440y_{n+2})] - 5(482x_{n+2} - 440y_{n+2})$

- $962x_{5n+6} - 40y_{5n+7} + 5[(962x_{3n+4} - 40y_{3n+5}) + 3(962x_{n+2} - 40y_{n+3})] - 5(962x_{n+2} - 40y_{n+3})$

- $\frac{1}{241}[(22x_{5n+7} - 9660y_{5n+5}) + 5[(22x_{3n+5} - 9660y_{3n+3}) + 3(22x_{n+3} - 9660y_{n+1})] - 5(22x_{n+3} - 9660y_{n+1})]$

- $\frac{1}{11}[(482x_{5n+7} - 9660y_{5n+6}) + 5[(482x_{3n+5} - 9660y_{3n+4}) + 3(482x_{n+3} - 9660y_{n+2})] - 5(482x_{n+3} - 9660y_{n+2})]$
- $10582x_{5n+7} - 9660y_{5n+7} + 5[(10582x_{3n+5} - 9660y_{3n+5}) + 3(10582x_{n+3} - 9660y_{n+3})] - 5(10582x_{n+3} - 9660y_{n+3})$

- $\frac{1}{6}[(11y_{5n+3} - 241y_{5n+5}) + 5[(11y_{3n+4} - 241y_{3n+3}) + 3(11y_{n+2} - 241y_{n+1})] - 5(11y_{n+2} - 241y_{n+1})]$

- $\frac{1}{12}[(y_{5n+7} - 481y_{5n+5}) + 5[(y_{3n+5} - 481y_{3n+3}) + 3(y_{n+3} - 481y_{n+1})] - 5(y_{n+3} - 481y_{n+1})]$

- $\frac{1}{6}[(241y_{5n+7} - 5291y_{5n+6}) + 5[(241y_{3n+5} - 5291y_{3n+4}) + 3(241y_{n+3} - 5291y_{n+2})] - 5(241y_{n+3} - 5291y_{n+2})]$

4.7 Remarkable Observations:

1. Employing linear combinations among the solutions of (4.1), one may generate integer solutions for other choices of hyperbolas which are presented in Table: 4.2 below.

Table:4.2 Hyperbolas

S.No	Hyperbolas	(P, Q)
1.	$30P^2 - Q^2 = 120$	$P = 44x_{n+1} - 2x_{n+2}$ $Q = 11x_{n+2} - 241x_{n+1}$
2.	$120P^2 - 121Q^2 = 58080$	$P = 483x_{n+1} - x_{n+3}$ $Q = x_{n+3} - 481x_{n+1}$
3.	$30P^2 - Q^2 = 120$	$P = 22x_{n+1} - 20y_{n+1}$ $Q = 110y_{n+1} - 120x_{n+1}$
4.	$30P^2 - 121Q^2 = 14520$	$P = 482x_{n+1} - 20y_{n+2}$ $Q = 10y_{n+2} - 240x_{n+1}$
5.	$30P^2 - Q^2 = 6969720$	$P = 10582x_{n+1} - 20y_{n+3}$ $Q = 110y_{n+3} - 57960x_{n+1}$
6.	$30P^2 - Q^2 = 120$	$P = 966x_{n+2} - 44x_{n+3}$ $Q = 241x_{n+3} - 5291x_{n+2}$

7.	$3630P^2 - Q^2 = 14520$	$P = 2x_{n+2} - 40y_{n+1}$ $Q = 2410y_{n+1} - 120x_{n+2}$
8.	$30P^2 - Q^2 = 120$	$P = 482x_{n+2} - 440y_{n+2}$ $Q = 2410y_{n+2} - 2640x_{n+2}$
9.	$3630P^2 - Q^2 = 14520$	$P = 962x_{n+2} - 40y_{n+3}$ $Q = 2410y_{n+3} - 57960x_{n+2}$
10.	$30P^2 - Q^2 = 6969720$	$P = 22x_{n+3} - 9660y_{n+1}$ $Q = 52910y_{n+1} - 120x_{n+3}$
11.	$30P^2 - Q^2 = 14520$	$P = 482x_{n+3} - 9660y_{n+2}$ $Q = 52910y_{n+2} - 2640x_{n+3}$
12.	$30P^2 - Q^2 = 120$	$P = 10582x_{n+3} - 9660y_{n+3}$ $Q = 52910y_{n+3} - 57960x_{n+3}$
13.	$30P^2 - 36Q^2 = 4320$	$P = 11y_{n+2} - 241y_{n+1}$ $Q = 220y_{n+1} - 10y_{n+2}$
14.	$14520P^2 - 144Q^2 = 8363520$	$P = y_{n+3} - 481y_{n+1}$ $Q = 4830y_{n+1} - 10y_{n+3}$
15.	$30P^2 - 36Q^2 = 4320$	$P = 241y_{n+3} - 5291y_{n+2}$ $Q = 4830y_{n+2} - 220y_{n+3}$

2. Employing linear combinations among the solutions of (4.1), one may generate integer solutions for other choices of parabolas which are presented in Table:4.3 below.

Table: 4.3 Parabolas

S.No	Parabolas	(R, Q)
1.	$30R - Q^2 = 120$	$R = 44x_{2n+2} - 2x_{2n+3} + 2$ $Q = 11x_{n+2} - 241x_{n+1}$
2.	$120R - 11Q^2 = 5280$	$R = 483x_{2n+2} - x_{2n+4} + 22$ $Q = x_{n+3} - 481x_{n+1}$
3.	$30R - Q^2 = 120$	$R = 22x_{2n+2} - 20y_{2n+2} + 2$ $Q = 110y_{n+1} - 120x_{n+1}$
4.	$30R - 11Q^2 = 1320$	$R = 482x_{2n+2} - 20y_{2n+3} + 22$ $Q = 10y_{n+2} - 240x_{n+1}$
5.	$7230R - Q^2 = 6969720$	$R = 10582x_{2n+2} - 20y_{2n+4} + 482$ $Q = 110y_{n+3} - 57960x_{n+1}$
6.	$30R - Q^2 = 120$	$R = 966x_{2n+3} - 44x_{2n+4} + 2$ $Q = 241x_{n+3} - 5291x_{n+2}$
7.	$3630R - Q^2 = 14520$	$R = 2x_{2n+3} - 40y_{2n+2} + 2$ $Q = 2410y_{n+1} - 120x_{n+2}$
8.	$30R - Q^2 = 120$	$R = 482x_{2n+3} - 440y_{2n+3} + 2$ $Q = 2410y_{n+2} - 2640x_{n+2}$
9.	$3630R - Q^2 = 14520$	$R = 962x_{2n+3} - 40y_{2n+4} + 2$ $Q = 2410y_{n+3} - 57960x_{n+2}$
10.	$7230R - Q^2 = 6969720$	$R = 22x_{2n+4} - 9660y_{2n+2} + 482$ $Q = 52910y_{n+1} - 120x_{n+3}$
11.	$3630R - Q^2 = 14520$	$R = 482x_{2n+4} - 9660y_{2n+3} + 22$ $Q = 52910y_{n+2} - 2640x_{n+3}$

12.	$30R - Q^2 = 120$	$R = 10582x_{2n+4} - 9660y_{2n+4} + 2$ $Q = 52910y_{n+3} - 57960x_{n+3}$
13.	$30R - 6Q^2 = 720$	$R = 11y_{2n+3} - 241y_{2n+2} + 12$ $Q = 220y_{n+1} - 10y_{n+2}$
14.	$14520R - 12Q^2 = 696960$	$R = y_{2n+4} - 481y_{2n+2} + 24$ $Q = 4830y_{n+1} - 10y_{n+3}$
15.	$30R - 6Q^2 = 720$	$R = 241y_{2n+4} - 5291y_{2n+3} + 12$ $Q = 4830y_{n+2} - 220y_{n+3}$

4.8 Conclusion

In this paper, we have presented infinitely many integer solutions for the second degree equation, represented by hyperbola is given by $6x^2 - 5y^2 = 6$. As the second degree equation are rich in variety, one may search for the other choices of equations and determine their integer solutions along with suitable properties.

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Chapter 5

A Glance on Finding Integer Solution To Hyperbola $5x^2 - 3y^2 = 18$

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Abstract: The non-homogenous binary quadratic equation with two unknowns represented by the Pell-like equation $5x^2 - 3y^2 = 18$ is studied for finding its distinct integer solutions. A few interesting properties between the above solutions are presented.

Keywords: Binary quadratic, Hyperbola, Parabola, Pell equation, Integral solutions.
2010 Mathematics subject classification: 11D09

5.1 Introduction

The non-homogenous binary quadratic equations of the form $ax^2 - by^2 = N, (a, b, N \neq 0)$ are rich in variety and have been analyzed by many mathematicians for their respective integer solutions for particular values of a,b and N. In this context, one may refer (Carmichael., 1959; Dickson., 1952; Mordell., 1969; Gopalan et.al., 2024; Mahalakshmi, Sowmiya .,2023; Mahalakshmi et.al.,2023;Shanthi,Gopalan.,2021). This communication concerns with the problem of obtaining non-zero distinct integer solutions to the binary quadratic equation given by$5x^2 - 3y^2 = 18$ representing hyperbola. A few interesting relations among its solutions are presented. Knowing an integral solutions of the given hyperbola, integer solution for other choices of hyperbolas and parabolas are presented.

5.2Method of analysis:

The non-homogenous binary quadratic equation under consideration is

$$5x^2 - 3y^2 = 18 \quad (5.1)$$

It is to be noted that (5.1) represents a hyperbola

Taking x=X+3T , y=X+5T (5.2)

in (5.1), it reduced to the equation

$$X^2 = 15T^2 + 9 \quad (5.3)$$

The smallest positive integer solution(T_0, X_0) of (5.3) is

$$T_0 = 3, X_0 = 12$$

To obtain the other solutions of (5.3), consider the pellian equations

$$X^2 = 15T^2 + 9 \quad (5.4)$$

whose smallest positive integer solutions is

$$\tilde{T}_0 = 1, \tilde{X}_0 = 4$$

The general solution $(\tilde{T}_n, \tilde{X}_n)$ of (5.4) is given by

$$\tilde{X}_n + \sqrt{15}\tilde{T}_n = (4 + \sqrt{15})^{n+1}, n = 0,1,2, \dots.. \quad (5.5)$$

Since irrational roots occur in pairs, we have

$$\tilde{X}_n - \sqrt{15}\tilde{T}_n = (4 - \sqrt{15})^{n+1}, n = 0,1,2, \dots. \quad (5.6)$$

From (5.5) and (5.6), solving for$\tilde{X}_n, \tilde{T}_n$, we have

$$\tilde{X}_n = \frac{1}{2}[(4 + \sqrt{15})^{n+1} + (4 - \sqrt{15})^{n+1}] = \frac{1}{2} f_n$$

$$\tilde{T}_n = \frac{1}{2\sqrt{15}}[(4 + \sqrt{15})^{n+1} - (4 - \sqrt{15})^{n+1}] = \frac{1}{2\sqrt{15}} g_n$$

Applying Brahmagupta lemma between the solutions (T_0, X_0) and $(\tilde{T}_n, \tilde{X}_n)$, the general solution $(\tilde{T}_{n+1}, \tilde{X}_{n+1})$ of (5.3) is found to be

$$T_{n+1} = X_0\tilde{T}_n + T_0\tilde{X}_n = \frac{6}{\sqrt{15}} g_n + \frac{3}{2} f_n \quad (5.7)$$

$$X_{n+1} = X_0\tilde{X}_n + T_0\tilde{T}_n = 6f_n + \frac{3}{2\sqrt{15}} g_n \quad (5.8)$$

Using (5.7) and (5.8) in (5.2) we have

$$10x_{n+1} = 105f_n + 27\sqrt{15}g_n \quad (5.9)$$

$$2y_{n+1} = 27f_n + 7\sqrt{15}g_n \quad (5.10)$$

Thus (5.9) and (5.10) represent the integer solution of the hyperbola (5.1).

A few numerical values are given in the following table:5.1

Table: 5.1 Numerical Examples

N	x_{n+1}	y_{n+1}
-1	21	27
0	165	213
1	1299	1677
2	10227	13203
3	80517	103947
4	633909	818373

Recurrence relations for x and y are:

$x_{n+3} - 8x_{n+2} + x_{n+1} = 0$, n= -1,0,1,...

$y_{n+3} - 8y_{n+2} + y_{n+1} = 0$, n=-1,0,1...

5.3 A few interesting relation among the solution are given below.

- $x_{n+1} - 8x_{n+2} + x_{n+3} = 0$
- $4x_{n+1} - x_{n+2} + 3y_{n+1} = 0$
- $x_{n+1} - 4x_{n+2} + 3y_{n+2} = 0$
- $4x_{n+1} - 31x_{n+2} + 3y_{n+3} = 0$
- $31x_{n+1} - x_{n+3} + 24y_{n+1} = 0$
- $x_{n+1} - x_{n+3} + 6y_{n+2} = 0$
- $x_{n+1} - 31x_{n+3} + 24y_{n+3} = 0$
- $y_{n+2} - 4y_{n+1} - 5x_{n+1} = 0$
- $y_{n+3} - 31y_{n+1} - 40x_{n+1} = 0$
- $4y_{n+3} - 5x_{n+1} - 31y_{n+2} = 0$
- $3y_{n+1} + 31x_{n+2} - 4x_{n+3} = 0$
- $3y_{n+2} + 4x_{n+2} - x_{n+3} = 0$
- $3y_{n+3} + x_{n+2} - 4x_{n+3} = 0$
- $3x_{n+3} - 4y_{n+1} - 31x_{n+2} = 0$

- $y_{n+3} - y_{n+1} - 10x_{n+2} = 0$
- $y_{n+1} + 5x_{n+2} - 4y_{n+2} = 0$
- $y_{n+3} - 4y_{n+2} - 5x_{n+2} = 0$
- $31y_{n+3} - y_{n+1} - 40x_{n+3} = 0$
- $4y_{n+3} - 5x_{n+3} - y_{n+2} = 0$
- $y_{n+3} + y_{n+1} - 8y_{n+2} = 0$

5.4 Each of following expressions represents a cubic integer

- $\frac{1}{3}(71x_{3n+3} - 9x_{3n+4}) + 3(71x_{n+1} - 9x_{n+2})$
- $\frac{1}{24}(559x_{3n+2} - 9x_{3n+5}) + 3(559x_{n+1} - 9x_{n+3})$
- $\frac{1}{3}(35x_{3n+3} - 27y_{3n+3}) + 3(35x_{n+1} - 27x_{n+3})$
- $\frac{1}{12}(275x_{3n+3} - 27y_{3n+4}) + 3(275x_{n+1} - 27y_{n+2})$
- $\frac{1}{93}(2165x_{3n+3} - 27y_{3n+5}) + 3(2165x_{n+1} - 27y_{n+3})$
- $\frac{1}{3}(559x_{3n+4} - 71x_{3n+5}) + 3(559x_{n+2} - 71x_{n+3})$
- $\frac{1}{12}(35x_{3n+4} - 213y_{3n+3}) + 3(35x_{n+2} - 213y_{n+1})$
- $\frac{1}{3}(275x_{3n+4} - 213y_{3n+4}) + 3(275x_{n+2} - 213y_{n+2})$
- $\frac{1}{12}(2165x_{3n+4} - 213y_{3n+5}) + 3(2165x_{n+2} - 213y_{n+3})$
- $\frac{1}{93}(35x_{3n+5} - 1677y_{3n+3}) + 3(35x_{n+3} - 1677y_{n+1})$
- $\frac{1}{12}(275x_{3n+5} - 1677y_{3n+4}) + 3(275x_{n+3} - 1677y_{n+2})$
- $\frac{1}{3}(2165x_{3n+5} - 1677y_{3n+5}) + 3(2165x_{n+3} - 1677y_{n+3})$
- $\frac{1}{3}(7y_{3n+4} - 55y_{3n+3}) + 3(7y_{n+2} - 55y_{n+1})$
- $\frac{1}{24}(7y_{3n+5} - 433y_{3n+3}) + 3(7y_{n+3} - 433y_{n+1})$
- $\frac{1}{3}(55y_{3n+5} - 433y_{3n+4}) + 3(55y_{n+3} - 433y_{n+2})$

5.5 Each of the following expressions represents a Bi-quadratic Integer

- $\frac{1}{3}[(71x_{4n+4} - 9x_{4n+5}) + 4(71x_{2n+2} - 9x_{2n+3})] + 6$
- $\frac{1}{24}[(559x_{4n+4} - 9x_{4n+9}) + 4(559x_{2n+2} - 9x_{2n+4})] + 6$
- $\frac{1}{3}[(35x_{4n+4} - 27y_{4n+4}) + 4(35x_{2n+2} - 27y_{2n+2})] + 6$
- $\frac{1}{12}[(275x_{4n+4} - 27y_{4n+5}) + 4(275x_{2n+2} - 27y_{2n+3})] + 6$

- $\frac{1}{93}[(2165x_{4n+4} - 27y_{4n+6}) + 4(2165x_{2n+2} - 27y_{2n+4})] + 6$
- $\frac{1}{3}[(559x_{4n+5} - 71x_{4n+6}) + 4(559x_{2n+3} - 71x_{2n+4})] + 6$
- $\frac{1}{12}[(35x_{4n+5} - 213y_{4n+4}) + 4(35x_{2n+3} - 213y_{2n+2})] + 6$
- $\frac{1}{3}[(275x_{4n+5} - 213y_{4n+5}) + 4(275x_{2n+3} - 213y_{2n+3})] + 6$
- $\frac{1}{12}[(2165x_{4n+5} - 213y_{4n+6}) + 4(2165x_{2n+3} - 213y_{2n+4})] + 6$
- $\frac{1}{93}[(35x_{4n+6} - 1677y_{4n+4}) + 4(35x_{2n+4} - 1677y_{2n+2})] + 6$
- $\frac{1}{12}[(275x_{4n+6} - 1677y_{4n+5}) + 4(275x_{2n+4} - 1677y_{2n+3})] + 6$
- $\frac{1}{3}[(2165x_{4n+6} - 1677y_{4n+6}) + 4(2165x_{2n+4} - 1677y_{2n+4})] + 6$
- $\frac{1}{3}[(7y_{4n+5} - 55y_{4n+4}) + 4(7y_{2n+3} - 55y_{2n+2})] + 6$
- $\frac{1}{24}[(7y_{4n+6} - 433y_{4n+4}) + 4(7y_{2n+4} - 433y_{2n+2})] + 6$
- $\frac{1}{3}[(55y_{4n+6} - 433y_{4n+5}) + 4(55y_{2n+4} - 433y_{2n+3})] + 6$

5.6 Each of the following expressions represents a Quintic Integer

- $\frac{1}{3}[(71x_{5n+5} - 9x_{5n+6}) + 5[(71x_{3n+3} - 9x_{3n+4}) + 3(71x_{n+1} - 9x_{n+2})] - 5(71x_{n+1} - 9x_{n+2})]$

- $\frac{1}{24}[(559x_{5n+5} - 9x_{5n+7}) + 5[(559x_{3n+3} - 9x_{3n+5}) + 3(559x_{n+1} - 9x_{n+3})] - 5(559x_{n+1} - 9x_{n+3})]$

- $\frac{1}{3}[(35x_{5n+5} - 27y_{5n+5}) + 5[(35x_{3n+3} - 27y_{3n+3}) + 3(35x_{n+1} - 27x_{n+3})] - 5(35x_{n+1} - 27y_{n+1})]$

- $\frac{1}{12}[(275x_{5n+5} - 27y_{5n+6}) + 5[(275x_{3n+3} - 27y_{3n+4}) + 3(275x_{n+1} - 27y_{n+2})] - 5(275x_{n+1} - 27y_{n+2})]$

- $\frac{1}{93}[(2165x_{5n+5} - 27y_{5n+7}) + 5[(2165x_{3n+3} - 27y_{3n+5}) + 3(2165x_{n+1} - 27y_{n+3})] - 5(2165x_{n+1} - 27y_{n+3})]$

- $\frac{1}{3}[(559x_{5n+6} - 71x_{5n+7}) + 5[(559x_{3n+4} - 71x_{3n+5}) + 3(559x_{n+2} - 71x_{n+3})] - 5(559x_{n+2} - 71x_{n+3})]$

- $\frac{1}{12}[(35x_{5n+6} - 213y_{5n+5}) + 5[(35x_{3n+4} - 213y_{3n+3}) + 3(35x_{n+2} - 213y_{n+1})] - 5(35x_{n+2} - 213y_{n+1})]$

- $\frac{1}{3}[(275x_{5n+6} - 213y_{5n+6}) + 5[(275x_{3n+4} - 213y_{3n+4}) + 3(275x_{n+2} - 213y_{n+2})] - 5(275x_{n+2} - 213y_{n+2})]$

- $\frac{1}{12}[(2165x_{5n+6} - 213y_{5n+7}) + 5[(2165x_{3n+4} - 213y_{3n+5}) + 3(2165x_{n+2} - 213y_{n+3})] - 5(2165x_{n+2} - 213y_{n+3})]$

- $\frac{1}{93}[(35x_{5n+7} - 1677y_{5n+5}) + 5[(35x_{3n+5} - 1677y_{3n+3}) + 3(35x_{n+3} - 1677y_{n+1})] - 5(35x_{n+3} - 1677y_{n+1})]$

- $\frac{1}{12}[(275x_{5n+7} - 1677y_{5n+6}) + 5[(275x_{3n+5} - 1677y_{3n+4}) + 3(275x_{n+3} - 1677y_{n+2})] - 5(275x_{n+3} - 1677y_{n+2})]$

- $\frac{1}{3}[(2165x_{5n+7} - 1677y_{5n+7}) + 5[(2165x_{3n+5} - 1677y_{3n+5}) + 3(2165x_{n+3} - 1677y_{n+3})] - 5(2165x_{n+3} - 1677y_{n+3})]$

- $\frac{1}{3}[(7y_{5n+6} - 55y_{5n+5}) + 5[(7y_{3n+4} - 55y_{3n+3}) + 3(7y_{n+2} - 55y_{n+1})] - 5(7y_{n+2} - 55y_{n+1})]$
- $\frac{1}{24}[(7y_{5n+7} - 433y_{5n+5}) + 5[(7y_{3n+5} - 433y_{3n+3}) + 3(7y_{n+3} - 433y_{n+1})] - 5(7y_{n+3} - 433y_{n+1})]$

- $\frac{1}{3}[(55y_{5n+7} - 433y_{5n+6}) + 5[(55y_{3n+5} - 433y_{3n+4}) + 3(55y_{n+3} - 433y_{n+2})] - 5(55y_{n+3} - 433y_{n+2})]$

5.7 Remarkable Observations:

1. Employing linear combinations among the solutions of (5.1), one may generate integer solutions for other choices of hyperbola which are presented in Table: 5.2 below.

Table:5.2 Hyperbolas

S.No	Hyperbolas	(P,Q)
1.	$30P^2 - 2Q^2 = 1080$	$P = (71x_{n+1} - 9x_{n+2})$ $Q = (35x_{n+2} - 275x_{n+1})$
2.	$1944000P^2 - 576Q^2 = 4478976000$	$P = (559x_{n+1} - 9x_{n+3})$ $Q = (525x_{n+3} - 32475x_{n+1})$
3.	$15P^2 - 9Q^2 = 540$	$P = (35x_{n+1} - 27y_{n+1})$ $Q = (35y_{n+1} - 45x_{n+1})$
4.	$240P^2 - 144Q^2 = 138240$	$P = (275x_{n+1} - 27y_{n+2})$ $Q = (35y_{n+2} - 355x_{n+1})$
5.	$14415P^2 - 8649Q^2 = 498701340$	$P = (2165x_{n+1} - 27y_{n+3})$ $Q = (35y_{n+3} - 2795x_{n+1})$
6.	$30P^2 - 2Q^2 = 1080$	$P = (559x_{n+2} - 71x_{n+3})$ $Q = (275x_{n+3} - 2165x_{n+2})$
7.	$240P^2 - 144Q^2 = 138240$	$P = (35x_{n+2} - 213y_{n+1})$ $Q = (275y_{n+1} - 45x_{n+2})$
8.	$15P^2 - 9Q^2 = 540$	$P = (275x_{n+2} - 213y_{n+2})$ $Q = (275y_{n+2} - 355x_{n+2})$
9.	$240P^2 - 144Q^2 = 138240$	$P = (2165x_{n+2} - 213y_{n+3})$ $Q = (275y_{n+3} - 2795x_{n+2})$
10.	$14415P^2 - 8649Q^2 = 498701340$	$P = (35x_{n+3} - 1677y_{n+1})$ $Q = (2165y_{n+1} - 45x_{n+3})$
11.	$240P^2 - 144Q^2 = 138240$	$P = (275x_{n+3} - 1677y_{n+2})$ $Q = (2165y_{n+2} - 355x_{n+3})$
12.	$15P^2 - 9Q^2 = 540$	$P = (2165x_{n+3} - 1677y_{n+3})$ $Q = (2165y_{n+3} - 2795x_{n+3})$
13.	$15P^2 - 9Q^2 = 540$	$P = (7y_{n+2} - 55y_{n+1})$ $Q = (71y_{n+1} - 9y_{n+2})$
14.	$960P^2 - 576Q^2 = 2211840$	$P = (7y_{n+3} - 433y_{n+1})$, $Q = (559y_{n+1} - 9y_{n+3})$
15.	$15P^2 - 9Q^2 = 540$	$P = (55y_{n+3} - 433y_{n+2})$ $Q = (559y_{n+2} - 71y_{n+3})$

2. Employing linear combinations among the solutions of (5.1), one may generate integer solutions for other choices of parabolas which are presented in Table: 5.3 below

Table: 5.3 Parabolas

S.No	Parabolas	(R,Q)
1.	$135R - 3Q^2 = 1620$	$R = (71x_{2n+2} - 9x_{2n+3} + 6)$ $Q = (35x_{n+2} - 275x_{n+1})$
2.	$1944000R - 24Q^2 = 186624000$	$R = (559x_{2n+2} - 9x_{2n+4} + 48)$ $Q = (525x_{n+3} - 32475x_{n+1})$
3.	$5R - Q^2 = 60$	$R = (35x_{2n+2} - 27y_{2n+2} + 6)$ $Q = (35y_{n+1} - 45x_{n+1})$
4.	$20R - Q^2 = 960$	$R = (275x_{2n+2} - 27y_{2n+3} + 24)$ $Q = (35y_{n+2} - 355x_{n+1})$
5.	$155R - Q^2 = 57660$	$R = (2165x_{2n+2} - 27y_{2n+4} + 186)$ $Q = (35y_{n+3} - 2795x_{n+1})$
6.	$135R - 3Q^2 = 1620$	$R = (559x_{2n+3} - 71x_{2n+4} + 6)$ $Q = (275x_{n+3} - 2165x_{n+2})$
7.	$20R - Q^2 = 960$	$R = (35x_{2n+3} - 213y_{2n+2} + 24)$ $Q = (275y_{n+1} - 45x_{n+2})$
8.	$5R - Q^2 = 60$	$R = (275x_{2n+3} - 213y_{2n+3} + 6)$ $Q = (275y_{n+2} - 355x_{n+2})$
9.	$20R - Q^2 = 960$	$R = (2165x_{2n+3} - 213y_{2n+4} + 24)$ $Q = (275y_{n+3} - 2795x_{n+2})$
10.	$155R - Q^2 = 57660$	$R = (35x_{2n+4} - 1677y_{2n+2} + 186)$ $Q = (2165y_{n+1} - 45x_{n+3})$
11.	$20R - Q^2 = 960$	$R = (275x_{2n+4} - 1677y_{2n+3} + 24)$ $Q = (2165y_{n+2} - 355x_{n+3})$
12.	$5R - Q^2 = 60$	$R = (2165x_{2n+4} - 1677y_{2n+4} + 6)$ $Q = (2165y_{n+3} - 2795x_{n+3})$
13.	$5R - Q^2 = 60$	$R = (7y_{2n+3} - 55y_{2n+2} + 6)$ $Q = (71y_{n+1} - 9y_{n+2})$
14.	$40R - Q^2 = 3840$	$R = (7y_{2n+4} - 433y_{2n+2} + 48)$ $Q = (559y_{n+1} - 9y_{n+3})$
15.	$5R - Q^2 = 60$	$R = (55y_{2n+4} - 433y_{2n+3} + 6)$ $Q = (559y_{n+2} - 71y_{n+3})$

5.8 Conclusion

In this paper, we have presented infinitely many integer solutions for the non-homogenous equation, represented by hyperbola is given $5x^2 - 3y^2 = 18$. As the non-homogeneous binary quadratic equations are rich in variety, one may search for the other choices of equations and determine their integer solutions along with suitable properties.

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Chapter 6

Choices of Solutions In Integers To Hyperbola $3x^2 - 2y^2 = 4$

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Abstract: The non-homogenous binary quadratic equation with two unknowns represented by the Pell-like equation $3x^2 - 2y^2 = 4$ is studied for finding its distinct integer solutions. A few interesting properties between the above solutions are presented.

Keywords: Binary quadratic, Hyperbola, Parabola, Pell equation, Integral solutions.
2010 Mathematics subject classification: 11D09

6.1 Introduction

The non-homogenous binary quadratic equations of the form $ax^2 - by^2 = N,(a,b,N \neq 0)$ are rich in variety and have been analyzed by many mathematicians for their respective integer solutions for particular values of a,b and N. In this context, one may refer (Carmichael., 1959; Dickson., 1952; Mordell., 1969; Mahalakshmi .,et.al .,2023; Shanthi.,et.al .,2023). This communication concerns with the problem of obtaining non-zero distinct integer solutions to the binary quadratic equation given by $3x^2 - 2y^2 = 4$ representing hyperbola. A few interesting relations among its solutions are presented. Knowing an integral solutions of the given hyperbola, integer solution for other choices of hyperbolas and parabolas are presented.

6.2Method of analysis:

The non-homogenous binary quadratic equation under consideration is

$3x2 - 2y^2 = 4$ (6.1)

It is to be noted that (6.1) represents a hyperbola

Taking x=X+2T,y=X+3T (6.2)

in (6.1), it reduced to the equation

$X^2 = 6T^2 + 4$ (6.3)

The smallest positive integer solution(T_0, X_0) of (6.3) is

$$T_0 = 4, X_0 = 10$$

To obtain the other solutions of (6.3), consider the pellian equations

$X^2 = 6T^2 + 1$ (6.4)

whose smallest positive integer solutions is

$$\tilde{T}_0 = 2, \tilde{X}_0 = 5$$

The general solution $(\tilde{T}_n, \tilde{X}_n)$ of (6.4) is given by

$\tilde{X}_n + \sqrt{6}\tilde{T}_n = (5 + 2\sqrt{6})^{n+1}, n = 0,1,2\ldots$ (6.5)

Since irrational roots occur in pairs, we have

$\tilde{X}_n - \sqrt{6}\tilde{T}_n = (5 - 2\sqrt{6})^{n+1}, n = 0,1,2\ldots$ (6.6)

From (6.5) and (6.6), solving for$\tilde{X}_n, \tilde{T}_n$, we have

$$\tilde{X}_n = \frac{1}{2}[(5 + 2\sqrt{6})^{n+1} + (5 - 2\sqrt{6})^{n+1}] = \frac{1}{2}f_n$$

$$\tilde{T}_n = \frac{1}{2\sqrt{6}}[(5 + 2\sqrt{6})^{n+1} - (5 - 2\sqrt{6})n + 1] = \frac{1}{2\sqrt{6}}g_n$$

Applying Brahmagupta lemma between the solutions (T_0, X_0) and $(\tilde{T}_n, \tilde{X}_n)$, the general solution $(\tilde{T}_{n+1}, \tilde{X}_{n+1})$ of (6.3) is found to be

$T_{n+1} = X_0\tilde{T}_n + T_0\tilde{X}_n = \frac{5}{\sqrt{6}}g_n + 2f_n$ (6.7)

$X_{n+1} = X_0\tilde{X}_n + T_0\tilde{T}_n = 2f_n + \frac{2}{\sqrt{6}}g_n$ (6.8)

Using (6.7) and (6.8) in (6.2) we have,

$$3x_{n+1} = 27f_n + 11\sqrt{6}g_n$$

$$2y_{n+1} = 22f_n + 9\sqrt{6}g_n$$

Thus (6.9) and (6.10) represent the integer of the hyperbola (6.1).

A few numerical values are given in the following Table: 6.1

Table: 6.1 Numerical Examples

n	x_{n+1}	y_{n+1}
-1	18	22
0	178	218
1	1762	2158
2	17442	21362

Recurrence relations for x and y are:

$$x_{n+3} - 10x_{n+2} + x_{n+1} = 0, n = -1,0,1\ldots..$$

$$y_{n+3} - 10y_{n+2} + y_{n+1} = 0, n = -1,0,1\ldots$$

6.3 A few interesting relations among the solutions are given be

- $x_{n+1} - 10x_{n+2} + x_{n+3} = 0$
- $5x_{n+1} - x_{n+2} + 4y_{n+1} = 0$
- $x_{n+1} - 5x_{n+2} + 4y_{n+2} = 0$
- $5x_{n+1} - 49x_{n+2} + 4y_{n+3} = 0$
- $49x_{n+1} - x_{n+3} + 40y_{n+1} = 0$
- $x_{n+1} - x_{n+3} + 8y_{n+2} = 0$
- $x_{n+1} - 49x_{n+3} + 40y_{n+3} = 0$
- $6x_{n+1} + 5y_{n+1} - y_{n+2} = 0$

- $60x_{n+1} + 49y_{n+1} - y_{n+3} = 0$
- $6x_{n+1} + 49y_{n+2} - 5y_{n+3} = 0$
- $4y_{n+1} + 49x_{n+2} - 5x_{n+3} = 0$
- $4y_{n+2} + 5x_{n+2} - x_{n+3} = 0$
- $4y_{n+3} + x_{n+2} - 5x_{n+3} = 0$
- $y_{n+1} + 6x_{n+2} - 5y_{n+2} = 0$
- $y_{n+1} + 12x_{n+2} - y_{n+3} = 0$
- $5y_{n+2} + 6x_{n+2} - y_{n+3} = 0$
- $49y_{n+2} - 5y_{n+1} - 6x_{n+3} = 0$
- $49y_{n+3} - y_{n+1} - 60x_{n+3} = 0$
- $5y_{n+3} - y_{n+2} - 6x_{n+3} = 0$
- $y_{n+3} - 10y_{n+2} + y_{n+1} = 0$

6.4 Each of following expressions represents a cubic integer

- $\frac{1}{2}(109x_{3n+3} - 11x_{3n+4}) + 3(109x_{n+1} - 11x_{n+2})$
- $\frac{1}{20}(1079x_{3n+3} - 11x_{3n+5}) + 3(1079x_{n+1} - 11x_{n+3})$
- $27x_{3n+3} - 22y_{3n+3} + 3(27x_{n+1} - 22y_{n+1})$
- $\frac{1}{5}(267x_{3n+3} - 22y_{3n+4}) + 3(267x_{n+1} - 22y_{n+2})$
- $\frac{1}{49}(2643x_{3n+3} - 22y_{3n+5}) + 3(2643x_{n+1} - 22y_{n+3})$
- $\frac{1}{2}(1079x_{3n+4} - 109x_{3n+5}) + 3(1079x_{n+2} - 109x_{n+3})$
- $\frac{1}{5}(27x_{3n+4} - 218y_{3n+3}) + 3(27x_{n+2} - 218y_{n+1})$
- $267x_{3n+4} - 218y_{3n+4} + 3(267x_{n+2} - 218y_{n+2})$

- $\frac{1}{5}(2643x_{3n+4} - 218y_{3n+5}) + 3(2643x_{n+2} - 218y_{n+3})$

- $\frac{1}{49}(27x_{3n+5} - 2158y_{3n+3}) + 3(27x_{n+3} - 2158y_{n+1})$

- $\frac{1}{5}(267_{3n+5} - 2158y_{3n+4}) + 3(267x_{n+3} - 2158y_{n+2})$

- $2643x_{3n+5} - 2158y_{3n+5} + 3(2643_{n+3} - 2158y_{n+3})$

- $\frac{1}{4}(18y_{3n+4} - 178y_{3n+3}) + 3(18y_{n+2} - 178y_{n+1})$

- $\frac{1}{40}(18y_{3n+5} - 1762y_{3n+3}) + 3(18y_{n+3} - 1762y_{n+1})$

- $\frac{1}{4}(178y_{3n+5} - 1762y_{3n+4}) + 3(178y_{n+3} - 1762y_{n+2})$

6.5 Each of following expressions represents a Bi-Quatratic integers

- $\frac{1}{2}[109x_{4n+4} - 11x_{4n+5} + 4(109x_{2n+2} - 11x_{2n+3})] + 6$

- $\frac{1}{20}[1079x_{4n+4} - 11x_{4n+6} + 4(1079x_{2n+2} - 11x_{2n+4})] + 6$

- $[27x_{4n+4} - 22x_{4n+4} + 4(27x_{2n+2} - 22x_{2n+2})] + 6$

- $\frac{1}{5}[267x_{4n+4} - 22x_{4n+5} + 4(267x_{2n+2} - 22x_{2n+3})] + 6$

- $\frac{1}{49}[2643x_{4n+4} - 22x_{4n+6} + 4(2643x_{2n+2} - 22x_{2n+4})] + 6$

- $\frac{1}{2}[1079x_{4n+5} - 109x_{4n+6} + 4(1079x_{2n+3} - 109x_{2n+4})] + 6$

- $[27x_{4n+5} - 218y_{4n+4} + 4(27x_{2n+2} - 11x_{2n+3})] + 6$

- $[267x_{4n+5} - 218y_{4n+5} + 4(267x_{2n+3} - 218y_{2n+2})] + 6$

- $[2643x_{4n+5} - 218y_{4n+6} + 4(2643x_{2n+3} - 218y_{2n+4})] + 6$

- $[27x_{4n+6} - 2158x_{4n+4} + 4(27x_{2n+4} - 2158y_{2n+2})] + 6$

- $[267x_{4n+6} - 2158y_{4n+5} + 4(267x_{2n+4} - 2158y_{2n+3})] + 6$

- $[2643x_{4n+6} - 2158y_{4n+6} + 4(2643x_{2n+4} - 2158y_{2n+4})] + 6$

- $[18y_{4n+5}-178y_{4n+4}+4(18y_{2n+3}-178y_{2n+2})]+6$
- $[18y_{4n+6}-1762y_{4n+4}+4(18y_{2n+4}-1762y_{2n+2})]+6$
- $[178y_{4n+6}-1762y_{4n+5}+4(178y_{2n+4}-1762y_{2n+3})]+6$

6.6 Each of following expressions represents a Quintic Integer

- $\frac{1}{2}[(109x_{5n+5}-11x_{5n+6}+5[(109x_{3n+3}-11x_{3n+4})+10(109x_{n+1}-11x_{n+2})]]$
- $\frac{1}{20}[(1079x_{5n+5}-11x_{5n+7}+5[(1079x_{3n+3}-11x_{3n+5})+10(1079x_{n+1}-11x_{n+3})]]$
- $[(27x_{5n+5}-22y_{5n+5}+5[(27x_{3n+3}-22y_{3n+3})+10(27x_{n+1}-22y_{n+1})]]$
- $\frac{1}{5}[(267x_{5n+5}-22y_{5n+6}+5[(267x_{3n+3}-22y_{3n+4})+10(267x_{n+1}-22y_{n+2})]]$
- $\frac{1}{49}[(2643x_{5n+5}-22y_{5n+7}+5[(2643x_{3n+3}-22y_{3n+5})+10(2643x_{n+1}-22y_{n+3})]]$
- $\frac{1}{2}[(1079x_{5n+6}-109x_{5n+7}+5[(1079x_{3n+4}-109x_{3n+5})+10(1079x_{n+2}-109x_{n+3})]]$
- $[(267x_{5n+6}-218y_{5n+6}+5[(267x_{3n+4}-218y_{3n+4})+10(267x_{n+2}-218y_{n+2})]]$
- $\frac{1}{5}[(2643x_{5n+6}-218y_{5n+7}+5[(2643x_{3n+4}-218y_{3n+5})+10(2643x_{n+2}-218y_{n+3})]]$
- $\frac{1}{49}[(27x_{5n+7}-2158y_{5n+5}+5[(27x_{3n+5}-2158y_{3n+5})+10(27x_{n+3}-2158y_{n+1})]]$
- $\frac{1}{5}[(267x_{5n+7}-2158y_{5n+6}+5[(267x_{3n+5}-2158y_{3n+4})+10(267x_{n+3}-2158y_{n+2})]]$

- $[(2643x_{5n+7} - 2158y_{5n+7} + 5[(2643x_{3n+5} - 2158y_{3n+5}) + 10(2643x_{n+3} - 2158y_{n+3})]]$

- $\frac{1}{4}[(18y_{5n+6} - 178y_{5n+5} + 5[(18y_{3n+4} - 178y_{3n+3}) + 10(18y_{n+2} - 178y_{n+1})]]$

- $\frac{1}{40}[(18y_{5n+7} - 1762y_{5n+5} + 5[(18y_{3n+5} - 1762y_{3n+3}) + 10(18y_{n+3} - 1762y_{n+1})]]$

- $\frac{1}{4}[(178y_{5n+7} - 1762y_{5n+6} + 5[(178y_{3n+5} - 1762y_{3n+4}) + 10(178y_{n+3} - 17621y_{n+2})]]$

- $\frac{1}{5}[(27x_{5n+6} - 218y_{5n+5} + 5[(27x_{3n+4} - 218y_{3n+3}) + 10(27x_{n+2} - 218y_{n+1})]]$

6.7 Remarkable Observations:

1.Employing linear combinations among the solutions of (6.1), one may generate integers solutions for other choices of hyperbolas which are presented in table : 6.2 below.

Table: 6.2 Hyperbolas

S.No	Hyperbolas	(P, Q)
1.	$6P^2 - Q^2 = 96$	$Q = (109x_{n+1} - 11x_{n+2})$ $P = (27x_{n+2} - 267x_{n+1})$
2.	$6P^2 - Q^2 = 9600$	$Q = (1079x_{n+1} - 11x_{n+3})$ $P = 3(9x_{n+3} - 881x_{n+1})$
3.	$6P^2 - 2Q^2 = 24$	$Q = (27x_{n+1} - 22y_{n+1})$ $P = (54y_{n+1} - 66x_{n+1})$
4.	$6P^2 - Q^2 = 600$	$Q = (267x_{n+1} - 22y_{n+2})$ $P = (54y_{n+2} - 654x_{n+1})$
5.	$6P^2 - Q^2 = 57624$	$Q = (2643x_{n+1} - 22y_{n+3})$ $P = (54y_{n+3} - 6474x_{n+1})$
6.	$6P^2 - Q^2 = 96$	$Q = (1079x_{n+2} - 109x_{n+3})$ $P = (267x_{n+3} - 2643x_{n+2})$
7.	$6P^2 - Q^2 = 600$	$Q = (27x_{n+2} - 218y_{n+1})$ $P = (534y_{n+1} - 66x_{n+2})$

8.	$6P^2 - Q^2 = 24$	$Q = (267x_{n+2} - 218y_{n+2})$ $P = (534y_{n+2} - 654x_{n+2})$
9.	$6P^2 - Q^2 = 600$	$Q = (2643x_{n+2} - 218y_{n+3})$ $P = (534y_{n+3} - 6474x_{n+2})$
10.	$6P^2 - Q^2 = 57624$	$Q = (27x_{n+3} - 2158y_{n+1})$ $P = (5286y_{n+1} - 66x_{n+3})$
11.	$6P^2 - Q^2 = 600$	$Q = (267x_{n+3} - 2158y_{n+2})$ $P = (5286y_{n+2} - 654x_{n+3})$
12.	$6P^2 - Q^2 = 24$	$Q = (2643x_{n+3} - 2158y_{n+3})$ $P = (5286y_{n+3} - 6474x_{n+3})$
13.	$6P^2 - Q^2 = 384$	$Q = (18y_{n+2} - 178y_{n+1})$ $P = (436y_{n+1} - 44y_{n+2})$
14.	$6P^2 - Q^2 = 6400$	$Q = (18y_{n+3} - 1762y_{n+1})$ $P = (4316y_{n+1} - 44y_{n+3})$
15.	$6P^2 - Q^2 = 384$	$Q = (178y_{n+3} - 1762y_{n+2})$ $P = (4316y_{n+2} - 436y_{n+3})$

2.Employing linear combination among the solutions of (6.1), one may generate integer solutions for other choices of parabolas which are presented in Table: 6.3 below:

Table: 6.3 Parabolas

S.No	Parabolas	(R, Q)
1.	$24R + 2Q^2 = 192$	$R = (109x_{2n+2} - 11x_{2n+3} + 4)$ $Q = (27x_{n+2} - 267x_{n+1})$
2.	$120R + Q^2 = 9600$	$R = (1079x_{2n+2} - 11x_{2n+4} + 40)$ $Q = (3(9x_{n+3} - 881x_{n+1})$
3.	$6R + Q^2 = 24$	$R = (27x_{2n+2} - 22y_{2n+2} + 2)$ $Q = (54y_{n+1} - 66x_{n+1})$
4.	$30R + Q^2 = 600$	$R = (267x_{2n+2} - 22y_{2n+3} + 10)$ $Q = (54y_{n+1} - 654x_{n+1})$
5.	$294R + Q^2 = 57624$	$R = (2643x_{2n+2} - 22y_{2n+4} + 98)$ $Q = (54y_{n+3} - 6474x_{n+1})$
6.	$12R + Q^2 = 96$	$R = (1079x_{2n+3} - 109x_{2n+4} + 4)$ $Q = (267x_{n+3} - 2643x_{n+2})]$
7.	$30R + Q^2 = 600$	$R = (27x_{2n+3} - 218y_{2n+2} + 10)$ $Q = (534y_{n+1} - 66x_{n+2})$
8.	$6R + Q^2 = 24$	$R = (267x_{2n+3} - 218y_{2n+3} + 2)$ $Q = (534y_{n+2} - 654x_{n+2})$

9.	$30R + Q^2 = 600$	$R = (2643x_{2n+3} - 218y_{2n+4} + 10)$ $Q = (534y_{n+3} - 6474x_{n+2})$
10.	$294R + Q^2 = 57624$	$R = (27x_{2n+4} - 2158y_{2n+2} + 98)$ $Q = (5286y_{n+1} - 66x_{n+3})$
11.	$30R + Q^2 = 600$	$R = (267x_{2n+4} - 2158y_{2n+3} + 10)$ $Q = (5286y_{n+2} - 654x_{n+3})$
12.	$6R + Q^2 = 24$	$R = (2643x_{2n+4} - 2158y_{2n+4} + 2)$ $Q = (5286y_{n+3} - 6474x_{n+3})$
13.	$24R + Q^2 = 384$	$R = (18y_{2n+3} - 178y_{2n+2} + 8)$ $Q = (436y_{n+1} - 44y_{n+2})$
14.	$240R + Q^2 = 38400$	$R = (18y_{2n+4} - 1762y_{2n+2} + 80)$ $Q = (4316y_{n+1} - 44y_{n+3})$
15.	$24R + Q^2 = 384$	$R = (178y_{2n+4} - 1762y_{2n+3} + 8)$ $Q = (4316y_{n+2} - 436y_{n+3})$

6.8 Conclusion

In this Paper, we have presented infinitely many integer solutions for the Non-homogeneous equations represented by hyperbola given by $5x^2 - 3y^2 = 18$. Non-homogeneous binary quadratic equations are rich in variety, one may search for the choices of equations and determine their integer solutions along with suitable properties.

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Chapter 7

On Homogeneous Ternary Quadratic Diophantine Equation $x^2 + y^2 = 125z^2$

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Abstract: The Quadratic Diophantine equation with three unknowns represented by $x^2 + y^2 = 125z^2$ is analyzed for finding its non-zero distinct integral solutions. Different patterns of solutions of the equation under consideration are obtained through factorization technique.

Keywords: Ternary quadratic equation, homogenous quadratic equation, integral solutions, factorization method, ratio form.

7.1 Introduction

The Quadratic Diophantine equation with three unknowns offers an unlimited field for research because of their variety (Carmichael., 1959; Dickson., 2005; Mordell., 1970). In particular, one may refer (Gopalan et.al .,2022;Vidhyalakshmi et.al,. 2021; Shanthi,Parkavi.,2023; Shanthi,Gopalan.,2024) for quadratic equations with three unknowns. This communication concerns with yet another interesting equation $x^2 + y^2 = 125z^2$ representing homogeneous Diophantine equation with three unknowns for determining its infinitely many non-zero integral solutions. A few interesting properties among its solutions are given.

7.2Method of analysis:

The ternary quadratic Diophantine equation to solved for its non-zero distinct integral solution is

$$x^2 + y^2 = 125z^2 \quad (7.1)$$

Different patterns of solution of (7.1) are presented below.

PATTERN -1

Equation (7.1) can be written as

$$x^2 + y^2 = 100z^2 + 25z^2$$

$$(x + 10z)(x - 10z) = (5z + y)(5z - y) \quad (7.2)$$

The process of solving (7.2) is illustrated as below:

Choice 1: Equation (7.2) can be written in ratio form as

$$\frac{x+10z}{5z+y} = \frac{5z-y}{x-10z} = \frac{\alpha}{\beta}, \quad \beta \neq 0$$

This equation is equal to the following two equations:

$$x\beta - y\alpha + (10\beta - 5\alpha)z = 0$$

$$-x\alpha - y\beta + (5\beta + 10\alpha)z = 0$$

By the method of cross multiplication, we get the integral solutions of (7.1) to be

$$x = 10\alpha^2 - 10\beta^2 + 10\alpha\beta$$

$$y = -5\alpha^2 + 5\beta^2 + 20\alpha\beta$$

$$z = \alpha^2 + \beta^2$$

Choice 2: Equation (7.2) can be written in ratio form as

$$\frac{x+10z}{5z-y} = \frac{5z+y}{x-10z} = \frac{\alpha}{\beta}, \quad \beta \neq 0$$

This equation is equal to the following two equations:

$$x\beta + y\alpha + (10\beta - 5\alpha)z = 0$$

$$-x\alpha + y\beta + (5\beta + 10\alpha)z = 0$$

By the method of cross multiplication, we get the integral solutions of (7.1) to be

$$x = 10\alpha^2 - 10\beta^2 + 10\alpha\beta$$
$$y = 5\alpha^2 - 5\beta^2 - 20\alpha\beta$$
$$z = \alpha^2 + \beta^2$$

Choice 3: Equation (7.2) can be written in ratio form as

$$\frac{x-10z}{5z+y} = \frac{5z-y}{x+10z} = \frac{\alpha}{\beta} \quad ,\beta \neq 0$$

This equation is equal to the following two equations:

$$x\beta - y\alpha - (10\beta + 5\alpha)z = 0$$
$$-x\alpha - y\beta + (5\beta - 10\alpha)z = 0$$

By the method of cross multiplication, we get the integral solutions of (7.1) to be

$$x = 10\alpha^2 - 10\beta^2 - 10\alpha\beta$$
$$y = 5\alpha^2 - 5\beta^2 + 20\alpha\beta$$
$$z = -\alpha^2 - \beta^2$$

Choice 4: Equation (7.2) can be written in ratio form as

$$\frac{x-10z}{5z-y} = \frac{5z+y}{x+10z} = \frac{\alpha}{\beta} \quad ,\beta \neq 0$$

This equation is equal to the following two equations:

$$x\beta + y\alpha - (10\beta + 5\alpha)z = 0$$
$$-x\alpha + y\beta + (5\beta - 10\alpha)z = 0$$

By the method of cross multiplication, we get the integral solutions of (7.1) to be

$$x = -10\alpha^2 + 10\beta^2 + 10\alpha\beta$$
$$y = 5\alpha^2 - 5\beta^2 + 20\alpha\beta$$
$$z = \alpha^2 + \beta^2$$

PATTERN-2

Equation (7.1) can be written as

$$x^2 + y^2 = 121z^2 + 4z^2$$

$$(x + 11z)(x - 11z) = (2z + y)(2z - y) \qquad (7.3)$$

The process of solving (7.3) is illustrated as below:

Choice 5: Equation (7.3) can be written in ratio form as

$$\frac{x+11z}{2z+y} = \frac{2z-y}{x-11z} = \frac{\alpha}{\beta} \;, \quad \beta \neq 0$$

This equation is equal to the following two equations:

$$x\beta - y\alpha + (11\beta - 2\alpha)z = 0$$
$$-x\alpha - y\beta + (2\beta + 11\alpha)z = 0$$

By the method of cross multiplication, we get the integral solutions of (7.1) to be

$$x = 11\alpha^2 - 11\beta^2 + 4\alpha\beta$$
$$y = -2\alpha^2 + 2\beta^2 + 22\alpha\beta$$
$$z = \alpha^2 + \beta^2$$

Choice 6: Equation (7.3) can be written in ratio form as

$$\frac{x+11z}{2z-y} = \frac{2z+y}{x-11z} = \frac{\alpha}{\beta} \;, \quad \beta \neq 0$$

This equation is equal to the following two equations:

$$x\beta + y\alpha + (11\beta - 2\alpha)z = 0$$
$$-x\alpha + y\beta + (2\beta + 11\alpha)z = 0$$

By the method of cross multiplication, we get the integral solutions of (7.1) to be

$$x = 11\alpha^2 - 11\beta^2 + 4\alpha\beta$$
$$y = 2\alpha^2 - 2\beta^2 - 22\alpha\beta$$
$$z = \alpha^2 + \beta^2$$

Choice 7: Equation (7.3) can be written in ratio form as

$\frac{x-11z}{2z+y} = \frac{2z-y}{x+11z} = \frac{\alpha}{\beta}$, $\beta \neq 0$

This equation is equal to the following two equations:

$$x\beta - y\alpha - (11\beta + 2\alpha)z = 0$$
$$-x\alpha - y\beta + (2\beta - 11\alpha)z = 0$$

By the method of cross multiplication, we get the integral solutions of (7.1) to be

$$x = 11\alpha^2 - 11\beta^2 - 4\alpha\beta$$
$$y = 2\alpha^2 - 2\beta^2 + 22\alpha\beta$$
$$z = -\alpha^2 - \beta^2$$

Choice 8: Equation (7.3) can be written in ratio form as

$\frac{x-11z}{2z-y} = \frac{2z+y}{x+11z} = \frac{\alpha}{\beta}$, $\beta \neq 0$

This equation is equal to the following two equations:

$$x\beta + y\alpha - (11\beta + 2\alpha)z = 0$$
$$-x\alpha + y\beta + (2\beta - 11\alpha)z = 0$$

By the method of cross multiplication, we get the integral solutions of (7.1) to be

$$x = -11\alpha^2 + 11\beta^2 + 4\alpha\beta$$
$$y = 2\alpha^2 - 2\beta^2 + 22\alpha\beta$$
$$z = \alpha^2 + \beta^2$$

PATTERN-3

Assume

$$z = z(a,b) = a^2 + b^2 \tag{7.4}$$

$$where\ a, b \neq 0,$$

Write

$$125 = 11^2 + 2^2 = (11 + 2i)(11 - 2i) \tag{7.5}$$

Substituting (7.4) and (7.5) in (7.1) and employing factorization. Consider

$$x + iy = (11 + 2i)(a + ib)^2$$

Equating the real and imaginary parts in the above equation, we acquire the integer solution to (7.1) as

$$x = 11a^2 - 11b^2 - 4ab$$
$$y = 2a^2 - 2b^2 + 22ab$$
$$z = a^2 + b^2$$

Also, we can write

$$125 = 2^2 + 11^2 = (2 + 11i)(2 - 11i) \quad (7.6)$$

Substituting (7.4) and (7.6) in (7.1) and employing the development of factorization. Write

$$x + iy = (2 + 11i)(a + ib)^2$$

Equating the real and imaginary parts in the top of the equation, we acquire

$$x = 2a^2 - 2b^2 - 22ab$$
$$y = 11a^2 - 11b^2 + 4ab$$
$$z = a^2 + b^2$$

PATTERN-4

Assume

$$z = z(a, b) = a^2 + b^2$$

$$where a, b \neq 0,$$

Write

$$125 = 10^2 + 5^2 = (10 + 5i)(10 - 5i) \quad (7.7)$$

Substituting (7.4) and (7.7) in (7.1) and employing the development of factorization. Write

$$x + iy = (10 + 5i)(a + ib)^2$$

Equating the real and imaginary parts in the top of the equation, we acquire

$$x = 10a^2 - 10b^2 - 10ab$$
$$y = 5a^2 - 5b^2 + 20ab$$
$$z = a^2 + b^2$$

We can also write

$$125 = 5^2 + 10^2 = (5 + 10i)(5 - 10i) \quad (7.8)$$

Substituting (7.4) and (7.8) in (7.1) and employing the development of factorization. Write

$$x + iy = (5 + 10i)(a + ib)^2$$

Equating the real and imaginary parts in the top of the equation, we acquire

$$x = 5a^2 - 5b^2 - 20ab$$
$$y = 10a^2 - 10b^2 + 10ab$$
$$z = a^2 + b^2$$

PATTERN-5

Equation (7.1) can be written as

$$x^2 = 125z^2 - y^2 = \left(\sqrt{125}z + y\right)\left(\sqrt{125}z - y\right) \quad (7.9)$$

Assume

$$x^2 = 125a^2 - b^2 \quad (7.10)$$

$$where a, b \neq 0,$$

Using (7.10) in (7.9) and applying method, we have

$$\sqrt{125}z + y = \left(\sqrt{125}a + b\right)^2$$

$$= 125a^2 + b^2 + 2\sqrt{125}ab$$

Equating the rational and irrational factors, we get the integer solution to (7.1) as

$$x = 125a^2 - b^2$$
$$y = 125a^2 + b^2$$
$$z = 2ab$$

PATTERN-6

Consider (7.1) as

$$x^2 + y^2 = 125z^2 * 1 \qquad (7.11)$$

$$(x + iy)(x - iy) = (10 + 5i)(10 - 5i)(a + ib)^2(a - ib)^2 * 1$$

Now, $x^2 + y^2 = z^2$

$$\Rightarrow \frac{(x+iy)(x-iy)}{z^2} = 1 \qquad (7.12)$$

Let us take

$$x = 2mn$$
$$y = m^2 - n^2$$
$$z = m^2 + n^2$$

Equation (7.12) can be written as

$$\frac{\left(2mn+i(m^2-n^2)\right)\left(2mn-i(m^2-n^2)\right)}{m^2+n^2} = 1 \qquad (7.13)$$

Substituting (7.13) in (7.11) and employing the technique of factorization,

write

$$x + iy = (10 + 5i)(a + ib)^2 \frac{\left(2mn+i(m^2-n^2)\right)}{m^2+n^2} \qquad (7.14)$$

We have that

$$(a + ib)^2 = a^2 - b^2 + i2ab \qquad (7.15)$$

Consider

$$(a + ib)^2 = f(a, b) + ig(a, b)$$

We have

$$f(a,b) = a^2 - b^2$$
$$g(a,b) = 2ab$$

Also, but

$$F(m,n) = 2mn$$
$$G(m,n) = m^2 - n^2$$

Substituting (7.15) in (7.14), we have

$$x + iy = \frac{1}{m^2 + n^2}[(10 + 5i)(f(a,b) + ig(a,b))(F(m,n) + i(G(m,n))]$$

$$x + iy = \frac{1}{m^2 + n^2}[10(fF - gG) + i10(gF + fG) + 5i(fF - gG) - 5(gF + fG)]$$

For integer solutions replace a by$(m^2 + n^2)P$and b by $(m^2 + n^2)Q$

Then, equating the real and imaginary parts , we get the integer solution to (7.1) as

$$x = m^2 + n^2[10(f(P,Q)F - g(P,Q)G) - 5(g(P,Q)F + f(P,Q)G)]$$
$$y = m^2 + n^2[10(g(P,Q)F - f(P,Q)G) + 5(f(P,Q)F - g(P,Q)G)]$$
$$z = (m^2 + n^2)^2(P^2 + Q^2)$$

PATTERN-8

Equation (7.11) can also be written as

$$(x + iy)(x - iy) = (5 + 10i)(5 - 10i)(a + ib)^2(a - ib)^2 * 1 \quad (7.16)$$

Consider

$$x + iy = (5 + 10i)(a + ib)^2 \frac{\left(2mn + i(m^2 - n^2)\right)}{m^2 + n^2} \quad (7.17)$$

By using (7.15)

$$\Rightarrow x+iy=\frac{1}{m^2+n^2}\Big[(5+10i)\big(f(a,b)+ig(a,b)\big)\Big(F(m,n)+i\big(G(m,n)\big)\Big)\Big]$$

$$\Rightarrow x+iy=\frac{1}{m^2+n^2}[5(fF-gG)+i5(gF+fG)+10i(fF-gG)-10(gF+fG)]$$

For integer solutions replace a by $(m^2+n^2)P$and b by $(m^2+n^2)Q$

Then, equating the real and imaginary parts, we get the integer solution to (1) as

$$x=m^2+n^2[5(f(P,Q)F-g(P,Q)G)-10(g(P,Q)F+f(P,Q)G)]$$
$$y=m^2+n^2[5(g(P,Q)F-f(P,Q)G)+10(f(P,Q)F-g(P,Q)G)]$$
$$z=(m^2+n^2)^2(P^2+Q^2)$$

PATTERN-9

Equation (7.11) can also be written as

$$(x+iy)(x-iy)=(2+11i)(2-11i)(a+ib)^2(a-ib)^2+1 \qquad (7.18)$$

consider

$$x+iy=(2+11i)(a+ib)^2\frac{\big(2mn+i(m^2-n^2)\big)}{m^2+n^2} \qquad (7.19)$$

$$\Rightarrow x+iy=\frac{1}{m^2+n^2}[(2+11i)(f(a,b)+ig(a,b))(F(m,n)+i(G(m,n))]$$

$$\Rightarrow x+iy=\frac{1}{m^2+n^2}[2(fF-gG)+i2(gF+fG)+11i(fF-gG)-11(gF+fG)]$$

For integer solution replace a by $(m^2+n^2)P$and b by $(m^2+n^2)Q$

Then, Equating the real and imaginary parts , we get

$$x = m^2 + n^2[2(f(P,Q)F - g(P,Q)G) - 11(g(P,Q)F + f(P,Q)G)]$$
$$y = m^2 + n^2[2(g(P,Q)F - f(P,Q)G) + 11(f(P,Q)F - g(P,Q)G)]$$
$$z = (m^2 + n^2)^2(P^2 + Q^2)$$

PATTERN-10

Equation (7.11) can also be written as

$$(x + iy)(x - iy) = (11 + 2i)(11 - 2i)(a + ib)^2(a - ib)^2 + 1 \quad (7.20)$$

consider

$$x + iy = (11 + 2i)(a + ib)^2 \frac{\left(2mn + i(m^2 - n^2)\right)}{m^2 + n^2} \quad (7.21)$$

$$\Rightarrow x + iy = \frac{1}{m^2 + n^2}[(11 + 2i)(f(a,b) + ig(a,b))(F(m,n) + i(G(m,n))]$$

$$\Rightarrow x + iy = \frac{1}{m^2 + n^2}[11(fF - gG) + i11(gF + fG) + 2i(fF - gG) - 2(gF + fG)]$$

For integer solution replace a by $(m^2 + n^2)P$and b by $(m^2 + n^2)Q$

Then, Equating the real and imaginary parts , we get

$$x = m^2 + n^2[11(f(P,Q)F - g(P,Q)G) - 2(g(P,Q)F + f(P,Q)G)]$$
$$y = m^2 + n^2[11(g(P,Q)F - f(P,Q)G) + 2(f(P,Q)F - g(P,Q)G)]$$
$$z = (m^2 + n^2)^2(P^2 + Q^2)$$

7.3 Conclusion:

The ternary quadratic Diophantine equations are prosperous in diversity. One possibly will search for further choices of Diophantine equations to discover their consequent integer solutions.

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Chapter 8

Designs of integer solutions to homogeneous ternary quadratic equation $x^2 + y^2 = 185z^2$

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Abstract: The Quadratic Diophantine equation with three unknowns represented by $x^2 + y^2 = 185z^2$ is analyzed for finding its non-zero distinct integral solutions. Different patterns of solutions of the equation under consideration are obtained through factorization technique.

Keywords: Ternary quadratic equation, homogenous quadratic equation, integral solutions, factorization method, ratio form.

8.1 Introduction

The Quadratic Diophantine equation with three unknowns offers an unlimited field for research because of their variety (Carmichael., 1959; Dickson., 2005; Mordell., 1970). In particular, one may refer (Gopalan et.al .,2015;Vidhyalakshmi et.al,. 2014; ; Shanthi et.al .,2014) for quadratic equations with three unknowns. This communication concerns with yet another interesting equation $x^2 + y^2 = 185z^2$ representing homogeneous Diophantine equation with three unknowns for determining its infinitely many non-zero integral solutions. A few interesting properties among its solutions are given.

8.2Method of analysis

The ternary quadratic Diophantine equation to solved for its non-zero distinct integral solution is

$$x^2 + y^2 = 185z^2 \tag{8.1}$$

Different patterns of solution of (8.1) are presented below.

PATTERN -1

Equation (8.1) can be written as

$$x^2 + y^2 = 169z^2 + 16z^2$$

$$(x + 13z)(x - 13z) = (4z + y)(4z - y) \tag{8.2}$$

The process of solving (8.2) is illustrated as below:

Choice 1: Equation (8.2) can be written in ratio form as

$$\frac{x+13z}{4z+y} = \frac{4z-y}{x-13z} = \frac{\alpha}{\beta} \quad , \beta \neq 0$$

This equation is equal to the following two equations:

$$\beta x - \alpha y + (13\beta - 4\alpha)z = 0$$

$$-\alpha x - \beta y + (4\beta + 13\alpha)z = 0$$

By the method of cross multiplication, we get the integral solutions of (8.1) to be

$$x = 13\alpha^2 - 13\beta^2 + 8\alpha\beta$$
$$y = -4\alpha^2 + 4\beta^2 + 26\alpha\beta$$
$$z = \alpha^2 + \beta^2$$

Choice 2: Equation (8.2) can be written in ratio form as

$$\frac{x+13z}{4z-y} = \frac{4z+y}{x-13z} = \frac{\alpha}{\beta}, \ \beta \neq 0$$

This equation is equal to the following two equations:

$$\beta x + \alpha y + (13\beta - 4\alpha)z = 0$$

$$-\alpha x + \beta y + (4\beta + 13\alpha)z = 0$$

By the method of cross multiplication, we get the integral solutions of (8.1) to be

$$x = 13\alpha^2 + 13\beta^2 - 8\alpha\beta$$
$$y = -4\alpha^2 + 4\beta^2 + 26\alpha\beta$$
$$z = -\alpha^2 - \beta^2$$

Choice 3: Equation (8.2) can be written in ratio as

$$\frac{x-13z}{4z+y} = \frac{4z-y}{x+13z} = \frac{\alpha}{\beta} \quad ,\beta \neq 0$$

This equation is equal to the following two equations:

$$\beta x - \alpha y - (13\beta + 4\alpha)z = 0$$

$$-\alpha x - \beta y - (13\alpha - 4\beta)z = 0$$

By the method of cross multiplication, we get the integral solutions of (8.1) to be

$$x = -13\alpha^2 + 13\beta^2 + 8\alpha\beta$$
$$y = -4\alpha^2 + 4\beta^2 - 26\alpha\beta$$
$$z = \alpha^2 + \beta^2$$

Choice 4: Equation (8.2) can be written in ratio form as

$$\frac{x-13z}{4z-y} = \frac{4z+y}{x+13z} = \frac{\alpha}{\beta} \quad ,\beta \neq 0$$

This equation is equal to the following two equations:

$$\beta x + \alpha y - (13\beta + 4\alpha)z = 0$$

$$-\alpha x + \beta y - (13\alpha - 4\beta)z = 0$$

By the method of cross multiplication, we get the integral solutions of (8.1) to be

$$x = 13\alpha^2 - 13\beta^2 - 8\alpha\beta$$
$$y = -4\alpha^2 + 4\beta^2 - 26\alpha\beta$$
$$z = -\alpha^2 - \beta^2$$

PATTERN-2

Equation (8.1) can be written as

$$x^2 + y^2 = 64z^2 + 121z^2$$

$$(x + 8z)(x - 8z) = (11z + y)(11z - y) \quad (8.3)$$

The process of solving (8.3) is illustrated as below:

Choice 5: Equation (8.3) can be written in ratio form as

$$\frac{x+8z}{11z+y} = \frac{11z-y}{x-8z} = \frac{\alpha}{\beta} \ , \quad \beta \neq 0$$

This equation is equal to the following two equations:

$$\beta x - \alpha y + (8\beta - 11\alpha)z = 0$$

$$-\alpha x - \beta y + (11\beta + 8\alpha)z = 0$$

By the method of cross multiplication, we get the integral solutions of (8.1) to be

$$x = 8\alpha^2 - 8\beta^2 + 22\alpha\beta$$
$$y = -11\alpha^2 + 11\beta^2 + 16\alpha\beta$$
$$z = \alpha^2 + \beta^2$$

Choice 6: Equation (8.3) can be written in ratio form as

$$\frac{x+8z}{11z-y} = \frac{11z+y}{x-8z} = \frac{\alpha}{\beta} \ , \quad \beta \neq 0$$

This equation is equal to the following two equations:

$$\beta x + \alpha y + (8\beta - 11\alpha)z = 0$$

$$-\alpha x + \beta y + (11\beta + 8\alpha)z = 0$$

By the method of cross multiplication, we get the integral solutions of (8.1) to be

$$x = -8\alpha^2 + 8\beta^2 - 22\alpha\beta$$
$$y = -11\alpha^2 + 11\beta^2 + 16\alpha\beta$$
$$z = -\alpha^2 - \beta^2$$

Choice 7: Equation (8.3) can be written in ratio form as

$$\frac{x-8z}{11z+y}=\frac{11z-y}{x+8z}=\frac{\alpha}{\beta}\ ,\qquad \beta\neq 0$$

This equation is equal to the following two equations:

$$\beta x-\alpha y-(8\beta+11\alpha)z=0$$

$$-\alpha x-\beta y+(11\beta-8\alpha)z=0$$

By the method of cross multiplication, we get the integral solutions of (8.1) to be

$$x=-8\alpha^2+8\beta^2+22\alpha\beta$$
$$y=-11\alpha^2+11\beta^2-16\alpha\beta$$
$$z=\alpha^2+\beta^2$$

Choice 8: Equation (8.3) can be written in ratio form as

$$\frac{x-8z}{11z-y}=\frac{11z+y}{x+8z}=\frac{\alpha}{\beta}\ ,\qquad \beta\neq 0$$

This equation is equal to the following two equations:

$$\beta x-\alpha y+(8\beta+11\alpha)z=0$$

$$-\alpha x-\beta y+(11\beta-8\alpha)z=0$$

By the method of cross multiplication, we get the integral solutions of (8.1) to be

$$x=8\alpha^2-8\beta^2-22\alpha\beta$$
$$y=-11\alpha^2+11\beta^2-16\alpha\beta$$
$$z=-\alpha^2-\beta^2$$

PATTERN-3

Assume

$$z=z(a,b)=a^2+b^2 \qquad (8.4)$$

$$where\ a,b\neq 0,$$

Write

$$185 = 13^2 + 4^2 = (13 + 4i)(13 - 4i) \quad (8.5)$$

Substituting (8.4) and (8.5) in (8.1) and employing factorization. Consider

$$x + iy = (13 + 4i)(a + ib)^2$$

Equating the real and imaginary parts in the above equation, we acquire the integer solution to (8.1) as

$$x = 13a^2 - 13b^2 - 8ab$$
$$y = 4a^2 - 4b^2 + 26ab$$
$$z = a^2 + b^2$$

Also, we can write

$$185 = 4^2 + 13^2 = (4 + 13i)(4 - 13i) \quad (8.6)$$

Substituting (8.4) and (8.6) in (8.1) and employing the development of factorization. Write

$$x + iy = (4 + 13i)(a + ib)^2$$

Equating the real and imaginary parts in the top of the equation, we acquire

$$x = 4a^2 - 4b^2 - 26ab$$
$$y = 13a^2 - 13b^2 + 8ab$$
$$z = a^2 + b^2$$

PATTERN-4

Assume

$$z = z(a, b) = a^2 + b^2$$

$$where a, b \neq 0,$$

Write

$$185 = 11^2 + 8^2 = (11 + 8i)(11 - 8i) \quad (8.7)$$

Substituting (8.4) and (8.7) in (8.1) and employing the development of factorization. Write

$$x + iy = (11 + 8i)(a + ib)^2$$

Equating the real and imaginary parts in the top of the equation, we acquire

$$x = 11a^2 - 11b^2 - 16ab$$
$$y = 8a^2 - 8b^2 + 22ab$$
$$z = a^2 + b^2$$

We can also write

$$185 = 8^2 + 11^2 = (8 + 11i)(8 - 11i) \quad (8.8)$$

Substituting (8.4) and (8.8) in (8.1) and employing the development of factorization. Write

$$x + iy = (8 + 11i)(a + ib)^2$$

Equating the real and imaginary parts in the top of the equation, we acquire

$$x = 8a^2 - 8b^2 - 22ab$$
$$y = 11a^2 - 11b^2 + 16ab$$
$$z = a^2 + b^2$$

PATTERN-5

Equation (8.1) can be written as

$$x^2 = 185z^2 - y^2 = \left(\sqrt{185}z + y\right)\left(\sqrt{185}z - y\right) \quad (8.9)$$

Assume

$$x^2 = 185a^2 - b^2 \quad (8.10)$$

$$where a, b \neq 0,$$

Using (8.10) in (8.9) and applying method, we have

$$\sqrt{185}z + y = \left(\sqrt{185}a + b\right)^2$$

$$= 185a^2 + b^2 + 2\sqrt{185}ab$$

Equating the rational and irrational factors, we get the integer solution to (8.1) as

$$x = 185a^2 - b^2$$
$$y = 185a^2 + b^2$$
$$z = 2ab$$

PATTERN-6

Consider (8.1) as

$$x^2 + y^2 = 185z^2 * 1 \qquad (8.11)$$

$$(x + iy)(x - iy) = (13 + 4i)(13 - 4i)(a + ib)^2(a - ib)^2 * 1$$

Now, $x^2 + y^2 = z^2$

$$\Rightarrow \frac{(x+iy)(x-iy)}{z^2} = 1 \qquad (8.12)$$

Let us take

$$x = 2mn$$
$$y = m^2 - n^2$$
$$z = m^2 + n^2$$

Equation (8.12) can be written as

$$\frac{\left(2mn+i(m^2-n^2)\right)\left(2mn-i(m^2-n^2)\right)}{m^2+n^2} = 1 \qquad (8.13)$$

Substituting (8.13) in (8.11) and employing the technique of factorization,

write

$$x + iy = (13 + 4i)(a + ib)^2 \frac{\left(2mn+i(m^2-n^2)\right)}{m^2+n^2} \quad (8.14)$$

We have that

$$(a + ib)^2 = a^2 - b^2 + i2ab \quad (8.15)$$

Consider

$$(a + ib)^2 = f(a, b) + ig(a, b)$$

We have

$$f(a,b) = a^2 - b^2$$
$$g(a,b) = 2ab$$

Also, but

$$F(m,n) = 2mn$$
$$G(m,n) = m^2 - n^2$$

Substituting (8.15) in (8.14), we have

$$x + iy = \frac{1}{m^2 + n^2}[(13 + 4i)(f(a,b) + ig(a,b))(F(m,n) + i(G(m,n))]$$

$$x + iy = \frac{1}{m^2 + n^2}[13(fF - gG) + i13(gF + fG) + 4i(fF - gG) - 4(gF - fG)]$$

For integer solutions replace a by$(m^2 + n^2)P$and b by $(m^2 + n^2)Q$

Then, equating the real and imaginary parts , we get the integer solution to (1) as

$$x = m^2 + n^2[13(f(P,Q)F - g(P,Q)G) - 4(g(P,Q)F + f(P,Q)G)]$$
$$y = m^2 + n^2[13(g(P,Q)F - f(P,Q)G) + 4(f(P,Q)F - g(P,Q)G)]$$
$$z = (m^2 + n^2)^2(P^2 + Q^2)$$

PATTERN-8

Equation (8.11) can also be written as

$$(x + iy)(x - iy) = (4 + 13i)(4 - 13i)(a + ib)^2(a - ib)^2 * 1 \qquad (8.16)$$

Consider

$$x + iy = (4 + 13i)(a + ib)^2 \frac{\left(2mn + i\left(m^2 - n^2\right)\right)}{m^2 + n^2} \qquad (8.17)$$

By using (15)

$$\Rightarrow x + iy = \frac{1}{m^2 + n^2}\Big[(4 + 13i)\big(f(a, b) + ig(a, b)\big)\Big(F(m, n) + i\big(G(m, n)\big)\Big)\Big]$$

$$\Rightarrow x + iy = \frac{1}{m^2 + n^2}[4(fF - gG) + i4(gF + fG) + 13i(fF - gG) - 13(gF + fG)]$$

For integer solutions replace a by $(m^2 + n^2)P$and b by $(m^2 + n^2)Q$

Then, equating the real and imaginary parts, we get the integer solution to (1) as

$$x = m^2 + n^2[4(f(P, Q)F - g(P, Q)G) - 13(g(P, Q)F + f(P, Q)G)]$$
$$y = m^2 + n^2[4(g(P, Q)F - f(P, Q)G) + 13(f(P, Q)F - g(P, Q)G)]$$
$$z = (m^2 + n^2)^2(P^2 + Q^2)$$

PATTERN-9

Equation (8.11) can also be written as

$$(x + iy)(x - iy) = (11 + 8i)(11 - 8i)(a + ib)^2(a - ib)^2 + 1 \quad (8.18)$$

consider

$$x + iy = (11 + 8i)(a + ib)^2 \frac{\big(2mn + i(m^2 - n^2)\big)}{m^2 + n^2} \quad (8.19)$$

$$\Rightarrow x + iy = \frac{1}{m^2 + n^2}[(11 + 8i)(f(a, b) + ig(a, b))(F(m, n) + i(G(m, n))]$$

$$\Rightarrow x + iy = \frac{1}{m^2 + n^2}[11(fF - gG) + i11(gf + fg) + 8i(fF - gG) - 8(gF + fG)]$$

For integer solution replace a by $(m^2 + n^2)P$and b by $(m^2 + n^2)Q$

Then, Equating the real and imaginary parts , we get

$$x = m^2 + n^2[11(f(P, Q)F - g(P, Q)G) - 8(g(P, Q)F + f(P, Q)G)]$$
$$y = m^2 + n^2[11(g(P, Q)F - f(P, Q)G) + 8(f(P, Q)F - g(P, Q)G)]$$

$z = (m^2 + n^2)^2 (P^2 + Q^2)$

PATTERN-10

Equation (8.11) can also be written as

$$(x + iy)(x - iy) = (8 + 11i)(8 - 11i)(a + ib)^2 (a - ib)^2 + 1 \quad (8.20)$$

consider

$$x + iy = (8 + 11i)(a + ib)^2 \frac{\left(2mn + i(m^2 - n^2)\right)}{m^2 + n^2} \quad (8.21)$$

$$\Rightarrow x + iy = \frac{1}{m^2 + n^2}[(8 + 11i)(f(a,b) + ig(a,b))(F(m,n) + i(G(m,n))]$$

$$\Rightarrow x + iy = \frac{1}{m^2 + n^2}[8(fF - gG) + i8(gF + fG) + 11i(fF - gG) - 11i(gF + fG)]$$

For integer solution replace a by $(m^2 + n^2)P$and b by $(m^2 + n^2)Q$

Then, Equating the real and imaginary parts , we get

$$x = m^2 + n^2[8(f(P,Q)F - g(P,Q)G) - 11(g(P,Q)F + f(P,Q)G)]$$
$$y = m^2 + n^2[8(g(P,Q)F - f(P,Q)G) + 11(f(P,Q)F - g(P,Q)G)]$$
$$z = (m^2 + n^2)^2 (P^2 + Q^2)$$

8.3 Conclusion:

The ternary quadratic Diophantine equations are prosperous in diversity. One possibly will search for further choices of Diophantine equations to discover their consequent integer solutions.

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Chapter 9

Patterns on integer solutions to homogeneous ternary quadratic equation $x^2 + y^2 = 85z^2$

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Abstract: The Quadratic Diophantine equation with three unknowns represented by $x^2 + y^2 = 85z^2$ is analyzed for finding its non-zero distinct integral solutions. Different patterns of solutions of the equation under consideration are obtained through factorization technique.

Keywords: Ternary quadratic equation, homogenous quadratic equation, integral solutions, factorization method, ratio form

9.1 Introduction

The Quadratic Diophantine equation with three unknowns offers an unlimited field for research because of their variety (Carmichael., 1959; Dickson., 2005; Mordell., 1970). In particular, one may refer (Gopalan et.al .,2015,2016;Vidhyalakshmi et.al,. 2014; ; Shanthi et.al .,2014) for quadratic equations with three unknowns. This communication concerns with yet another interesting equation $x^2 + y^2 = 85z^2$ representing homogeneous Diophantine equation with three unknowns for determining its infinitely many non-zero integral solutions. A few interesting properties among its solutions are given.

9.2 Method of analysis

The ternary quadratic Diophantine equation to solved for its non-zero distinct integral solution is

$x^2 + y^2 = 85z^2$ (9.1)

Different patterns of solution of (9.1) are presented below.

PATTERN -1

Equation (9.1) can be written as

$$x^2 + y^2 = 81z^2 + 4z^2$$

$(x + 9z)(x - 9z) = (2z + y)(2z - y)$ (9.2)

The process of solving (9.2) is illustrated as below:

Choice 1: Equation (9.2) can be written in ratio form as

$\frac{x+9z}{2z+y} = \frac{2z-y}{x-9z} = \frac{\alpha}{\beta}$, $\beta \neq 0$

This equation is equal to the following two equations:

$$x\beta - y\alpha + (9\beta - 2\alpha)z = 0$$
$$-x\alpha - y\beta + (2\beta + 9\alpha)z = 0$$

By the method of cross multiplication, we get the integral solutions of (9.1) to be

$$x = 9\alpha^2 - 9\beta^2 + 4\alpha\beta$$
$$y = -2\alpha^2 + 2\beta^2 + 18\alpha\beta$$
$$z = \alpha^2 + \beta^2$$

Choice 2: Equation (9.2) can be written in ratio form as

$\frac{x+9z}{2z-y} = \frac{2z+y}{x-9z} = \frac{\alpha}{\beta}$, $\beta \neq 0$

This equation is equal to the following two equations:

$$x\beta + y\alpha + (9\beta - 2\alpha)z = 0$$
$$-x\alpha + y\beta + (9\beta + 2\alpha)z = 0$$

By the method of cross multiplication, we get the integral solutions of (9.1) to be

$$x = -9\alpha^2 + 9\beta^2 - 4\alpha\beta$$
$$y = -2\alpha^2 + 2\beta^2 + 18\alpha\beta$$
$$z = -\alpha^2 - \beta^2$$

Choice 3: Equation (9.2) can be written in ratio as

$\frac{x-9z}{2z+y} = \frac{2z-y}{x+9z} = \frac{\alpha}{\beta} \quad ,\beta \neq 0$

This equation is equal to the following two equations:

$$x\beta - y\alpha - (9\beta + 2\alpha)z = 0$$
$$-x\alpha - y\beta + (2\beta - 9\alpha)z = 0$$

By the method of cross multiplication, we get the integral solutions of (9.1) to be

$$x = -9\alpha^2 + 9\beta^2 + 4\alpha\beta$$
$$y = -2\alpha^2 + 2\beta^2 - 18\alpha\beta$$
$$z = \alpha^2 + \beta^2$$

Choice 4: Equation (9.2) can be written in ratio form as

$\frac{x-9z}{2z-y} = \frac{2z+y}{x+9z} = \frac{\alpha}{\beta} \quad ,\beta \neq 0$

This equation is equal to the following two equations:

$$\beta x + \alpha y - (9\beta + 2\alpha)z = 0$$

$$-\alpha x + \beta y - (9\alpha - 2\beta)z = 0$$

By the method of cross multiplication, we get the integral solutions of (9.1) to be

$$x = 9\alpha^2 - 9\beta^2 - 4\alpha\beta$$
$$y = -2\alpha^2 + 2\beta^2 - 18\alpha\beta$$
$$z = -\alpha^2 - \beta^2$$

PATTERN-2

Equation (9.1) can be written as

$$x^2 + y^2 = 49z^2 + 36z^2$$

$$(x + 7z)(x - 7z) = (6z + y)(6z - y) \tag{9.3}$$

The process of solving (9.3) is illustrated as below:

Choice 5: Equation (9.3) can be written in ratio form as

$$\frac{x+7z}{6z+y} = \frac{6z-y}{x-7z} = \frac{\alpha}{\beta} \quad , \qquad \beta \neq 0$$

This equation is equal to the following two equations:

$$\beta x - \alpha y + (7\beta - 6\alpha)z = 0$$

$$-\alpha x - \beta y + (6\beta + 7\alpha)z = 0$$

By the method of cross multiplication, we get the integral solutions of (9.1) to be

$$x = 7\alpha^2 - 7\beta^2 + 12\alpha\beta$$
$$y = -6\alpha^2 + 6\beta^2 + 14\alpha\beta$$
$$z = \alpha^2 + \beta^2$$

Choice 6: Equation (9.3) can be written in ratio form as

$$\frac{x+7z}{6z-y} = \frac{6z+y}{x-7z} = \frac{\alpha}{\beta} \quad , \qquad \beta \neq 0$$

This equation is equal to the following two equations:

$$\beta x + \alpha y + (7\beta - 6\alpha)z = 0$$

$$-\alpha x + \beta y + (6\beta + 7\alpha)z = 0$$

By the method of cross multiplication, we get the integral solutions of (9.1) to be

$$x = -7\alpha^2 + 7\beta^2 - 12\alpha\beta$$
$$y = -6\alpha^2 + 6\beta^2 + 14\alpha\beta$$
$$z = -\alpha^2 - \beta^2$$

Choice 7: Equation (9.3) can be written in ratio form as

$$\frac{x-7z}{6z+y} = \frac{6z-y}{x+7z} = \frac{\alpha}{\beta} \quad , \qquad \beta \neq 0$$

This equation is equal to the following two equations:

$$\beta x - \alpha y - (7\beta + 6\alpha)z = 0$$

$$-\alpha x - \beta y + (6\beta - 7\alpha)z = 0$$

By the method of cross multiplication, we get the integral solutions of (9.1) to be

$$x = -7\alpha^2 + 7\beta^2 + 12\alpha\beta$$
$$y = -6\alpha^2 + 6\beta^2 - 14\alpha\beta$$
$$z = \alpha^2 + \beta^2$$

The process of solving (9.6) is illustrated as below:

Choice 8: Equation (9.3) can be written in ratio form as

$$\frac{x-7z}{6z-y} = \frac{6z+y}{x+7z} = \frac{\alpha}{\beta}\ , \qquad \beta \neq 0$$

This equation is equal to the following two equations:

$$\beta x - \alpha y - (7\beta + 6\alpha)z = 0$$

$$-\alpha x - \beta y + (6\beta - 7\alpha)z = 0$$

By the method of cross multiplication, we get the integral solutions of (9.1) to be

$$x = 7\alpha^2 - 7\beta^2 + 12\alpha\beta$$
$$y = -6\alpha^2 + 6\beta^2 - 14\alpha\beta$$
$$z = -\alpha^2 - \beta^2$$

PATTERN-3

Assume

$$z = z(a,b) = a^2 + b^2 \qquad (9.4)$$

$$where\ a,b \neq 0,$$

Write

$$85 = 9^2 + 2^2 = (9 + 2i)(9 - 2i) \tag{9.5}$$

Substituting (9.4) and (9.5) in (9.1) and employing factorization. Consider

$$x + iy = (9 + 2i)(a + ib)^2$$

Equating the real and imaginary parts in the above equation, we acquire the integer solution to (9.1) as

$$x = 9a^2 - 9b^2 - 4ab$$
$$y = 2a^2 - 2b^2 + 18ab$$
$$z = a^2 + b^2$$

Also, we can write

$$85 = 2^2 + 9^2 = (2 + 9i)(2 - 9i) \tag{9.6}$$

Substituting (9.4) and (9.6) in (9.1) and employing the development of factorization. Write

$$x + iy = (2 + 9i)(a + ib)^2$$

Equating the real and imaginary parts in the top of the equation, we acquire

$$x = 2a^2 - 2b^2 - 18ab$$
$$y = 9a^2 - 9b^2 + 4ab$$
$$z = a^2 + b^2$$

PATTERN-4

Assume

$$z = z(a, b) = a^2 + b^2$$

$$where a, b \neq 0,$$

Write

$$85 = 7^2 + 6^2 = (7 + 6i)(7 - 6i) \tag{9.7}$$

Substituting (9.4) and (9.7) in (9.1) and employing the development of factorization. Write

$$x + iy = (7 + 6i)(a + ib)^2$$

Equating the real and imaginary parts in the top of the equation, we acquire

$$x = 7a^2 - 7b^2 - 12ab$$
$$y = 6a^2 - 6b^2 + 14ab$$
$$z = a^2 + b^2$$

We can also write

$$85 = 6^2 + 7^2 = (6 + 7i)(6 - 7i) \quad (9.8)$$

Substituting (9.4) and (9.8) in (9.1) and employing the development of factorization. Write

$$x + iy = (6 + 7i)(a + ib)^2$$

Equating the real and imaginary parts in the top of the equation, we acquire

$$x = 6a^2 - 6b^2 - 14ab$$
$$y = 7a^2 - 7b^2 + 12ab$$
$$z = a^2 + b^2$$

PATTERN-5

Equation (9.1) can be written as

$$x^2 = 85z^2 - y^2 = \left(\sqrt{85}z + y\right)\left(\sqrt{85}z - y\right) \quad (9.9)$$

Assume

$$x^2 = 85a^2 - b^2 \quad (9.10)$$

$$where a, b \neq 0,$$

Using (9.10) in (9.9) and applying method, we have

$$\sqrt{85}z + y = \left(\sqrt{85}a + b\right)^2$$

$$= 85a^2 + b^2 + 2\sqrt{85}ab$$

Equating the rational and irrational factors, we get the integer solution to (9.1) as

$$x = 185a^2 - b^2$$
$$y = 185a^2 + b^2$$
$$z = 2ab$$

PATTERN-6

Consider (9.1) as

$$x^2 + y^2 = 85z^2 * 1 \qquad (9.11)$$

$$(x + iy)(x - iy) = (9 + 2i)(9 - 2i)(a + ib)^2(a - ib)^2 * 1$$

Now, $x^2 + y^2 = z^2$

$$\Rightarrow \frac{(x+iy)(x-iy)}{z^2} = 1 \qquad (9.12)$$

Let us take

$$x = 2mn$$
$$y = m^2 - n^2$$
$$z = m^2 + n^2$$

Equation (9.12) can be written as

$$\frac{\left(2mn+i(m^2-n^2)\right)\left(2mn-i(m^2-n^2)\right)}{m^2+n^2} = 1 \qquad (9.13)$$

Substituting (9.13) in (9.11) and employing the technique of factorization,

write

$$x + iy = (9 + 2i)(a + ib)^2 \frac{\left(2mn+i(m^2-n^2)\right)}{m^2+n^2} \qquad (9.14)$$

We have that

$$(a + ib)^2 = a^2 - b^2 + i2ab \qquad (9.15)$$

Consider

$$(a + ib)^2 = f(a, b) + ig(a, b)$$

We have

$$f(a,b) = a^2 - b^2$$
$$g(a,b) = 2ab$$

Also, but

$$F(m,n) = 2mn$$
$$G(m,n) = m^2 - n^2$$

Substituting (9.15) in (9.14), we have

$$x + iy = \frac{1}{m^2 + n^2}[(9 + 2i)(f(a,b) + ig(a,b))(F(m,n) + i(G(m,n))]$$

$$x + iy = \frac{1}{m^2 + n^2}[9(fF - gG) + i9(gF + fG) + 2i(fF - gG) - 2(gF - fG)]$$

For integer solutions replace a by$(m^2 + n^2)P$and b by $(m^2 + n^2)Q$

Then, equating the real and imaginary parts , we get the integer solution to (9.1) as

$$x = m^2 + n^2[9(f(P,Q)F - g(P,Q)G) - 2(g(P,Q)F + f(P,Q)G)]$$
$$y = m^2 + n^2[9(g(P,Q)F - f(P,Q)G) + 2(f(P,Q)F - g(P,Q)G)]$$
$$z = (m^2 + n^2)^2(P^2 + Q^2)$$

PATTERN-8

Equation (9.11) can also be written as

$$(x + iy)(x - iy) = (2 + 9i)(2 - 9i)(a + ib)^2(a - ib)^2 * 1 \tag{9.16}$$

Consider

$$x + iy = (2 + 9i)(a + ib)^2 \frac{\left(2mn + i(m^2 - n^2)\right)}{m^2 + n^2} \tag{9.17}$$

By using (9.15)

$$\Rightarrow x+iy=\frac{1}{m^2+n^2}\Big[(2+9i)\big(f(a,b)+ig(a,b)\big)\Big(F(m,n) + i\big(G(m,n)\big)\Big)\Big]$$

$$\Rightarrow x+iy=\frac{1}{m^2+n^2}[2(fF-gG)+i2(gF+fG)+9i(fF-gG) - 9(gF+fG)]$$

For integer solutions replace a by $(m^2+n^2)P$and b by $(m^2+n^2)Q$

Then, equating the real and imaginary parts, we get the integer solution to (9.1) as

$$x=m^2+n^2[2(f(P,Q)F-g(P,Q)G)-9(g(P,Q)F+f(P,Q)G)]$$
$$y=m^2+n^2[2(g(P,Q)F-f(P,Q)G)+9(f(P,Q)F-g(P,Q)G)]$$
$$z=(m^2+n^2)^2(P^2+Q^2)$$

PATTERN-9

Equation (9.11) can also be written as

$$(x+iy)(x-iy)=(7+6i)(7-6i)(a+ib)^2(a-ib)^2+1 \qquad (9.18)$$

consider

$$x+iy=(7+6i)(a+ib)^2\frac{\big(2mn+i(m^2-n^2)\big)}{m^2+n^2} \qquad (9.19)$$

$$\Rightarrow x+iy=\frac{1}{m^2+n^2}[(7+6i)(f(a,b)+ig(a,b))(F(m,n) + i(G(m,n))]$$

$$\Rightarrow x+iy=\frac{1}{m^2+n^2}[7(fF-gG)+i7(gf+fg)+6i(fF-gG) - 6(gF+fG)]$$

For integer solution replace a by $(m^2+n^2)P$and b by $(m^2+n^2)Q$

Then, Equating the real and imaginary parts , we get

$$x = m^2 + n^2[7(f(P,Q)F - g(P,Q)G) - 6(g(P,Q)F + f(P,Q)G)]$$
$$y = m^2 + n^2[7(g(P,Q)F - f(P,Q)G) + 6(f(P,Q)F - g(P,Q)G)]$$
$$z = (m^2 + n^2)^2(P^2 + Q^2)$$

PATTERN-10

Equation (9.11) can also be written as

$$(x + iy)(x - iy) = (6 + 7i)(6 - 7i)(a + ib)^2(a - ib)^2 + 1 \qquad (9.20)$$

consider

$$x + iy = (6 + 7i)(a + ib)^2 \frac{\left(2mn + i(m^2 - n^2)\right)}{m^2 + n^2} \qquad (9.21)$$

$$\Rightarrow x + iy = \frac{1}{m^2 + n^2}[(6 + 7i)(f(a,b) + ig(a,b))(F(m,n) + i(G(m,n))]$$

$$\Rightarrow x + iy = \frac{1}{m^2 + n^2}[6(fF - gG) + i6(gF + fG) + 7i(fF - gG) - 7i(gF + fG)]$$

For integer solution replace a by $(m^2 + n^2)P$and b by $(m^2 + n^2)Q$

Then, Equating the real and imaginary parts , we get

$$x = m^2 + n^2[6(f(P,Q)F - g(P,Q)G) - 7(g(P,Q)F + f(P,Q)G)]$$
$$y = m^2 + n^2[6(g(P,Q)F - f(P,Q)G) + 7(f(P,Q)F - g(P,Q)G)]$$
$$z = (m^2 + n^2)^2(P^2 + Q^2)$$

9.3 Conclusion:

The ternary quadratic Diophantine equations are prosperous in diversity. One possibly will search for further choices of Diophantine equations to discover their consequent integer solutions.

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Chapter 10

A scrutiny of integer solutions to non-homogeneous ternary quadratic equation $x^2 + y^2 = 250z^2$

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Abstract: The Quadratic Diophantine equation with three unknowns represented by $x^2 + y^2 = 250z^2$ is analyzed for finding its non-zero distinct integral solutions. Different patterns of solutions of the equation under consideration are obtained through factorization technique.

Keywords: Ternary quadratic equation, homogenous quadratic equation, integral solutions, factorization method, ratio form

10.1 Introduction

The Quadratic Diophantine equation with three unknowns offers an unlimited field for research because of their variety (Carmichael., 1959; Dickson., 2005; Mordell., 1970). In particular, one may refer (Vidhyalakshmi et.al,. 2021;Maheswari.et.al.,2020 ; Shanthi et.al .,2020,2023) for quadratic equations with three unknowns. This communication concerns with yet another interesting equation $x^2 + y^2 = 250z^2$ representing homogeneous Diophantine equation with three unknowns for determining its infinitely many non-zero integral solutions. A few interesting properties among its solutions are given.

10.2 Method of analysis

The ternary quadratic Diophantine equation to solved for its non-zero distinct integral solution is

$$x^2 + y^2 = 250z^2 \tag{10.1}$$

Different patterns of solution of (10.1) are presented below.

PATTERN -1

Equation (10.1) can be written as

$$x^2 + y^2 = 225z^2 + 25z^2$$

$$(x + 15z)(x - 15z) = (5z + y)(5z - y) \tag{10.2}$$

The process of solving (10.2) is illustrated as below:

Choice 1: Equation (10.2) can be written in ratio form as

$$\frac{x+15z}{5z+y} = \frac{5z-y}{x-15z} = \frac{\alpha}{\beta}\ , \ \beta \neq 0$$

This equation is equal to the following two equations:

$$x\beta - y\alpha + (15\beta - 5\alpha)z = 0$$
$$-x\alpha - y\beta + (5\beta + 15\alpha)z = 0$$

By the method of cross multiplication, we get the integral solutions of (10.1) to be

$$x = 15\alpha^2 - 15\beta^2 + 10\alpha\beta$$
$$y = -5\alpha^2 + 5\beta^2 + 30\alpha\beta$$
$$z = \alpha^2 + \beta^2$$

Choice 2: Equation (10.2) can be written in ratio form as

$$\frac{x+15z}{5z-y} = \frac{5z+y}{x-15z} = \frac{\alpha}{\beta}\ , \ \beta \neq 0$$

This equation is equal to the following two equations:

$$x\beta + y\alpha + (15\beta - 5\alpha)z = 0$$
$$-x\alpha + y\beta + (5\beta + 15\alpha)z = 0$$

By the method of cross multiplication, we get the integral solutions of (10.1) to be

$$x = 15\alpha^2 - 15\beta^2 + 10\alpha\beta$$
$$y = 5\alpha^2 - 5\beta^2 - 30\alpha\beta$$
$$z = \alpha^2 + \beta^2$$

Choice 3: Equation (10.2) can be written in ratio form as

$$\frac{x-15z}{5z+y} = \frac{5z-y}{x+15z} = \frac{\alpha}{\beta} \quad ,\beta \neq 0$$

This equation is equal to the following two equations:

$$x\beta - y\alpha - (15\beta + 5\alpha)z = 0$$
$$-x\alpha - y\beta + (5\beta - 15\alpha)z = 0$$

By the method of cross multiplication, we get the integral solutions of (10.1) to be

$$x = 15\alpha^2 - 15\beta^2 - 10\alpha\beta$$
$$y = 5\alpha^2 - 5\beta^2 + 30\alpha\beta$$
$$z = -\alpha^2 - \beta^2$$

Choice 4: Equation (10.2) can be written in ratio form as

$$\frac{x-15z}{5z-y} = \frac{5z+y}{x+15z} = \frac{\alpha}{\beta} \quad ,\beta \neq 0$$

This equation is equal to the following two equations:

$$x\beta + y\alpha - (15\beta + 5\alpha)z = 0$$
$$-x\alpha + y\beta + (5\beta - 15\alpha)z = 0$$

By the method of cross multiplication, we get the integral solutions of (10.1) to be

$$x = -15\alpha^2 + 15\beta^2 + 10\alpha\beta$$
$$y = 5\alpha^2 - 5\beta^2 + 30\alpha\beta$$
$$z = \alpha^2 + \beta^2$$

PATTERN-2

Equation (10.1) can be written as

$$x^2 + y^2 = 81z^2 + 169z^2$$

$$(x + 9z)(x - 9z) = (13z + y)(13z - y) \quad (10.3)$$

The process of solving (10.3) is illustrated as below:

Choice 5: Equation (10.3) can be written in ratio form as

$$\frac{x+9z}{13z+y} = \frac{13z-y}{x-9z} = \frac{\alpha}{\beta}\ , \quad \beta \neq 0$$

This equation is equal to the following two equations:

$$x\beta - y\alpha + (9\beta - 13\alpha)z = 0$$
$$-x\alpha - y\beta + (13\beta + 9\alpha)z = 0$$

By the method of cross multiplication, we get the integral solutions of (10.1) to be

$$x = 9\alpha^2 - 9\beta^2 + 26\alpha\beta$$
$$y = -13\alpha^2 + 13\beta^2 + 18\alpha\beta$$
$$z = \alpha^2 + \beta^2$$

Choice 6: Equation (10.3) can be written in ratio form as

$$\frac{x+9z}{13z-y} = \frac{13z+y}{x-9z} = \frac{\alpha}{\beta}\ , \quad \beta \neq 0$$

This equation is equal to the following two equations:

$$x\beta + y\alpha + (9\beta - 13\alpha)z = 0$$
$$-x\alpha + y\beta + (13\beta + 9\alpha)z = 0$$

By the method of cross multiplication, we get the integral solutions of (10.1) to be

$$x = 9\alpha^2 - 9\beta^2 + 26\alpha\beta$$
$$y = 13\alpha^2 - 13\beta^2 - 18\alpha\beta$$
$$z = \alpha^2 + \beta^2$$

Choice 7: Equation (10.3) can be written in ratio form as

$$\frac{x-9z}{13z+y} = \frac{13z-y}{x+9z} = \frac{\alpha}{\beta} \quad , \qquad \beta \neq 0$$

This equation is equal to the following two equations:

$$x\beta - y\alpha - (9\beta + 13\alpha)z = 0$$
$$-x\alpha - y\beta + (13\beta - 9\alpha)z = 0$$

By the method of cross multiplication, we get the integral solutions of (10.1) to be

$$x = 9\alpha^2 - 9\beta^2 - 26\alpha\beta$$
$$y = 13\alpha^2 - 13\beta^2 + 18\alpha\beta$$
$$z = -\alpha^2 - \beta^2$$

Choice 8: Equation (10.3) can be written in ratio form as

$$\frac{x-9z}{13z-y} = \frac{13z+y}{x+9z} = \frac{\alpha}{\beta} \quad , \qquad \beta \neq 0$$

This equation is equal to the following two equations:

$$x\beta + y\alpha - (9\beta + 13\alpha)z = 0$$
$$-x\alpha + y\beta + (13\beta - 9\alpha)z = 0$$

By the method of cross multiplication, we get the integral solutions of (10.1) to be

$$x = -9\alpha^2 + 9\beta^2 + 26\alpha\beta$$
$$y = 13\alpha^2 - 13\beta^2 + 18\alpha\beta$$
$$z = \alpha^2 + \beta^2$$

PATTERN-3

Assume

$$z = z(a, b) = a^2 + b^2 \qquad (10.4)$$

$$where \ a, b \neq 0,$$

Write

$$250 = 15^2 + 5^2 = (15 + 5i)(15 - 5i) \qquad (10.5)$$

Substituting (10.4) and (10.5) in (10.1) and employing factorization. Consider

$$x + iy = (15 + 5i)(a + ib)^2$$

Equating the real and imaginary parts in the above equation, we acquire the integer solution to (10.1) as

$$x = 15a^2 - 15b^2 - 10ab$$
$$y = 5a^2 - 5b^2 + 30ab$$
$$z = a^2 + b^2$$

Also, we can write

$$250 = 5^2 + 15^2 = (5 + 15i)(5 - 15i) \qquad (10.6)$$

Substituting (10.4) and (10.6) in (10.1) and employing the development of factorization. Write

$$x + iy = (5 + 15i)(a + ib)^2$$

Equating the real and imaginary parts in the top of the equation, we acquire

$$x = 5a^2 - 5b^2 - 30ab$$
$$y = 15a^2 - 15b^2 + 10ab$$
$$z = a^2 + b^2$$

PATTERN-4

Assume

$$z = z(a, b) = a^2 + b^2$$

$$where a, b \neq 0,$$

Write

$$250 = 9^2 + 13^2 = (9 + 13i)(9 - 13i) \qquad (10.7)$$

Substituting (10.4) and (10.7) in (10.1) and employing the development of factorization. Write

$$x + iy = (9 + 13i)(a + ib)^2$$

Equating the real and imaginary parts in the top of the equation, we acquire

$$x = 9a^2 - 9b^2 - 26ab$$
$$y = 13a^2 - 13b^2 + 18ab$$
$$z = a^2 + b^2$$

We can also write

$$250 = 13^2 + 9^2 = (13 + 9i)(13 - 9i) \quad (10.8)$$

Substituting (10.4) and (10.8) in (10.1) and employing the development of factorization. Write

$$x + iy = (13 + 9i)(a + ib)^2$$

Equating the real and imaginary parts in the top of the equation, we acquire

$$x = 13a^2 - 13b^2 - 18ab$$
$$y = 9a^2 - 9b^2 + 26ab$$
$$z = a^2 + b^2$$

PATTERN-5

Equation (10.1) can be written as

$$x^2 = 250z^2 - y^2 = \left(\sqrt{250}z + y\right)\left(\sqrt{250}z - y\right) \quad (10.9)$$

Assume

$$x^2 = 250a^2 - b^2 \quad (10.10)$$

$$where a, b \neq 0,$$

Using (10.10) in (10.9) and applying method, we have

$$\sqrt{250}z + y = \left(\sqrt{250}a + b\right)^2$$

$$= 250a^2 + b^2 + 2\sqrt{250}ab$$

Equating the rational and irrational factors, we get the integer solution to (10.1) as

$$x = 250a^2 - b^2$$
$$y = 250a^2 + b^2$$
$$z = 2ab$$

PATTERN-6

Consider (10.1) as

$$x^2 + y^2 = 250z^2 * 1 \tag{10.11}$$

$$(x + iy)(x - iy) = (15 + i5)(15 - i5)(a + ib)^2(a - ib)^2 * 1$$

Now, $x^2 + y^2 = z^2$

$$\Rightarrow \frac{(x+iy)(x-iy)}{z^2} = 1 \tag{10.12}$$

Let us take

$$x = 2mn$$
$$y = m^2 - n^2$$
$$z = m^2 + n^2$$

Equation (10.12) can be written as

$$\frac{\left(2mn+i\left(m^2-n^2\right)\right)\left(2mn-i\left(m^2-n^2\right)\right)}{m^2+n^2} = 1 \tag{10.13}$$

Substituting (10.13) in (10.11) and employing the technique of factorization,

write

$$x + iy = (15 + 5i)(a + ib)^2 \frac{\left(2mn+i\left(m^2-n^2\right)\right)}{m^2+n^2} \tag{10.14}$$

We have that

$$(a + ib)^2 = a^2 - b^2 + i2ab \tag{10.15}$$

Consider

$$(a + ib)^2 = f(a, b) + ig(a, b)$$

We have

$$f(a,b) = a^2 - b^2$$
$$g(a,b) = 2ab$$

Also, but

$$F(m,n) = 2mn$$
$$G(m,n) = m^2 - n^2$$

Substituting (10.15) in (10.14), we have

$$x + iy = \frac{1}{m^2 + n^2}[(15 + 5i)(f(a,b) + ig(a,b))(F(m,n) + i(G(m,n))]$$

$$x + iy = \frac{1}{m^2 + n^2}[15(fF - gG) + i15(gF + fG) + 5i(fF - gG) - 5(gF + fG)]$$

For integer solutions replace a by $(m^2 + n^2)P$and b by $(m^2 + n^2)Q$

Then, equating the real and imaginary parts, we get the integer solution to (10.1) as

$$x = m^2 + n^2[15(f(P,Q)F - g(P,Q)G) - 5(g(P,Q)F + f(P,Q)G)]$$
$$y = m^2 + n^2[15(g(P,Q)F - f(P,Q)G) + 5(f(P,Q)F - g(P,Q)G)]$$
$$z = (m^2 + n^2)^2(P^2 + Q^2)$$

PATTERN-8

Equation (11) can also be written as

$$(x + iy)(x - iy) = (5 + 15i)(5 - 15i)(a + ib)^2(a - ib)^2 * 1 \quad (10.16)$$

Consider

$$x + iy = (5 + 15i)(a + ib)^2 \frac{\left(2mn + i\left(m^2 - n^2\right)\right)}{m^2 + n^2} \quad (10.17)$$

By using (10.15)

$$\Rightarrow x + iy = \frac{1}{m^2 + n^2}\Big[(5 + 15i)\big(f(a, b) + ig(a, b)\big)\Big(F(m, n) + i\big(G(m, n)\big)\Big)\Big]$$

$$\Rightarrow x + iy = \frac{1}{m^2 + n^2}[5(fF - gG) + i5(gF + fG) + 15i(fF - gG) - 15(gF + fG)]$$

For integer solutions replace a by $(m^2 + n^2)P$and b by $(m^2 + n^2)Q$

Then, equating the real and imaginary parts, we get the integer solution to (1) as

$$x = m^2 + n^2[5(f(P, Q)F - g(P, Q)G) - 15(g(P, Q)F + f(P, Q)G)]$$
$$y = m^2 + n^2[5(g(P, Q)F - f(P, Q)G) + 15(f(P, Q)F - g(P, Q)G)]$$
$$z = (m^2 + n^2)^2(P^2 + Q^2)$$

PATTERN-9

Equation (10.11) can also be written as

$$(x + iy)(x - iy) = (9 + 13i)(9 - 13i)(a + ib)^2(a - ib)^2 + 1 \quad (10.18)$$

consider

$$x + iy = (9 + 13i)(a + ib)^2 \frac{\big(2mn + i(m^2 - n^2)\big)}{m^2 + n^2} \quad (10.19)$$

$$\Rightarrow x + iy = \frac{1}{m^2 + n^2}[(9 + 13i)(f(a, b) + ig(a, b))(F(m, n) + i(G(m, n))]$$

$$\Rightarrow x + iy = \frac{1}{m^2 + n^2}[9(fF - gG) + i9(gF + fG) + 13i(fF - gG) - 13(gF + fG)]$$

For integer solution replace a by $(m^2 + n^2)P$and b by $(m^2 + n^2)Q$

Then, Equating the real and imaginary parts , we get

$$x = m^2 + n^2[9(f(P,Q)F - g(P,Q)G) - 13(g(P,Q)F + f(P,Q)G)]$$
$$y = m^2 + n^2[9(g(P,Q)F - f(P,Q)G) + 13(f(P,Q)F - g(P,Q)G)]$$
$$z = (m^2 + n^2)^2(P^2 + Q^2)$$

PATTERN-10

Equation (10.11) can also be written as

$$(x + iy)(x - iy) = (13 + 9i)(13 - 9i)(a + ib)^2(a - ib)^2 + 1 \quad (10.20)$$

consider

$$x + iy = (13 + 9i)(a + ib)^2 \frac{\left(2mn + i(m^2 - n^2)\right)}{m^2 + n^2} \quad (10.21)$$

$$\Rightarrow x + iy = \frac{1}{m^2 + n^2}[(13 + 9i)(f(a,b) + ig(a,b))(F(m,n) + i(G(m,n))]$$

$$\Rightarrow x + iy = \frac{1}{m^2 + n^2}[13(fF - gG) + i13(gF + fG) + 9i(fF - gG) - 9(gF + fG)]$$

For integer solution replace a by $(m^2 + n^2)P$and b by $(m^2 + n^2)Q$

Then, Equating the real and imaginary parts, we get

$$x = m^2 + n^2[13(f(P,Q)F - g(P,Q)G) - 9(g(P,Q)F + f(P,Q)G)]$$
$$y = m^2 + n^2[13(g(P,Q)F - f(P,Q)G) + 9(f(P,Q)F - g(P,Q)G)]$$
$$z = (m^2 + n^2)^2(P^2 + Q^2)$$

10.3 Conclusion:

The ternary quadratic Diophantine equations are prosperous in diversity. One possibly will search for further choices of Diophantine equations to discover their consequent integer solutions.

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Chapter 11

A portrayal of integer solutions to homogeneous ternary quadratic equation $x^2 + y^2 = 130z^2$

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Abstract: The Quadratic Diophantine equation with three unknowns represented by $\boldsymbol{x^2 + y^2 = 130z^2}$ is analyzed for finding its non-zero distinct integral solutions. Different patterns of solutions of the equation under consideration are obtained through factorization technique.

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Keywords: Ternary quadratic equation, homogenous quadratic equation, integral solutions, factorization method, ratio form

11.1 Introduction

The Quadratic Diophantine equation with three unknowns offers an unlimited field for research because of their variety (Carmichael., 1959; Dickson., 2005; Mordell., 1970). In particular, one may refer (Vidhyalakshmi et.al,. 2020 ; Shanthi et.al .,2020,2021)for quadratic equations with three unknowns. This communication concerns with yet another interesting equation $\boldsymbol{x^2 + y^2 = 130z^2}$ representing homogeneous Diophantine equation with three unknowns for determining its infinitely many non-zero integral solutions. A few interesting properties among its solutions are given.

11.2 Method of analysis

The ternary quadratic Diophantine equation to solved for its non-zero distinct integral solution is

$$x^2 + y^2 = 130z^2 \tag{11.1}$$

Different patterns of solution of (11.1) are presented below.

PATTERN -1

Equation (11.1) can be written as

$$x^2 + y^2 = 121z^2 + 9z^2$$

$$(x + 11z)(x - 11z) = (3z + y)(3z - y) \tag{11.2}$$

The process of solving (11.2) is illustrated as below:

Choice 1: Equation (11.2) can be written in ratio form as

$$\frac{x+11z}{3z+y} = \frac{3z-y}{x-11z} = \frac{\alpha}{\beta} \quad , \beta \neq 0$$

This equation is equal to the following two equations:

$$\beta x - \alpha y + (11\beta - 3\alpha)z = 0$$

$$-\alpha x - \beta y + (3\beta + 11\alpha)z = 0$$

By the method of cross multiplication, we get the integral solutions of (11.1) to be

$$x = 11\alpha^2 - 11\beta^2 + 6\alpha\beta$$
$$y = -3\alpha^2 + 3\beta^2 + 22\alpha\beta$$
$$z = \alpha^2 + \beta^2$$

Choice 2: Equation (11.2) can be written in ratio form as

$$\frac{x+11z}{3z-y} = \frac{3z+y}{x-11z} = \frac{\alpha}{\beta}, \ \beta \neq 0$$

This equation is equal to the following two equations:

$$\beta x + \alpha y + (11\beta - 3\alpha)z = 0$$

$$-\alpha x+\beta y+(3\beta+11\alpha)z=0$$

By the method of cross multiplication, we get the integral solutions of (11.1) to be

$$x=11\alpha^2-11\beta^2+6\alpha\beta$$
$$y=3\alpha^2-3\beta^2-22\alpha\beta$$
$$z=\alpha^2+\beta^2$$

Choice 3: Equation (11.2) can be written in ratio as

$$\frac{x-11z}{3z+y}=\frac{3z-y}{x+11z}=\frac{\alpha}{\beta} \quad ,\beta\neq 0$$

This equation is equal to the following two equations:

$$\beta x-\alpha y-(11\beta+3\alpha)z=0$$

$$-\alpha x-\beta y+(3\beta-11\alpha)z=0$$

By the method of cross multiplication, we get the integral solutions of (11.1) to be

$$x=11\alpha^2-11\beta^2-6\alpha\beta$$
$$y=3\alpha^2-3\beta^2+22\alpha\beta$$
$$z=-\alpha^2-\beta^2$$

Choice 4: Equation (11.2) can be written in ratio form as

$$\frac{x-11z}{3z-y}=\frac{3z+y}{x+11z}=\frac{\alpha}{\beta} \quad ,\beta\neq 0$$

This equation is equal to the following two equations:

$$\beta x+\alpha y-(11\beta+3\alpha)z=0$$

$$-\alpha x+\beta y+(3\beta-11\alpha)z=0$$

By the method of cross multiplication, we get the integral solutions of (11.1) to be

$$x=-11\alpha^2+11\beta^2+6\alpha\beta$$
$$y=3\alpha^2-3\beta^2+22\alpha\beta$$
$$z=\alpha^2+\beta^2$$

PATTERN-2

Equation (11.1) can be written as

$$x^2 + y^2 = 49z^2 + 81z^2$$

$$(x + 7z)(x - 7z) = (9z + y)(9z - y) \quad (11.3)$$

The process of solving (11.3) is illustrated as below:

Choice 5: Equation (11.3) can be written in ratio form as

$$\frac{x+7z}{9z+y} = \frac{9z-y}{x-7z} = \frac{\alpha}{\beta} \quad , \qquad \beta \neq 0$$

This equation is equal to the following two equations:

$$\beta x - \alpha y + (7\beta - 9\alpha)z = 0$$

$$-\alpha x - \beta y + (9\beta + 7\alpha)z = 0$$

By the method of cross multiplication, we get the integral solutions of (11.1) to be

$$x = 7\alpha^2 - 7\beta^2 + 18\alpha\beta$$
$$y = -9\alpha^2 + 9\beta^2 + 14\alpha\beta$$
$$z = \alpha^2 + \beta^2$$

Choice 6: Equation (11.3) can be written in ratio form as

$$\frac{x+7z}{9z-y} = \frac{9z+y}{x-7z} = \frac{\alpha}{\beta} \quad , \qquad \beta \neq 0$$

This equation is equal to the following two equations:

$$\beta x + \alpha y + (7\beta - 9\alpha)z = 0$$

$$-\alpha x + \beta y + (9\beta + 7\alpha)z = 0$$

By the method of cross multiplication, we get the integral solutions of (11.1) to be

$$x = 7\alpha^2 - 7\beta^2 + 18\alpha\beta$$
$$y = 9\alpha^2 - 9\beta^2 - 14\alpha\beta$$
$$z = \alpha^2 + \beta^2$$

Choice 7: Equation (11.3) can be written in ratio form as

$\frac{x-7z}{9z+y} = \frac{9z-y}{x+7z} = \frac{\alpha}{\beta}$, $\beta \neq 0$

This equation is equal to the following two equations:

$$\beta x - \alpha y - (7\beta + 9\alpha)z = 0$$

$$-\alpha x - \beta y + (9\beta - 7\alpha)z = 0$$

By the method of cross multiplication, we get the integral solutions of (11.1) to be

$$x = 7\alpha^2 - 7\beta^2 - 18\alpha\beta$$
$$y = 9\alpha^2 - 9\beta^2 + 14\alpha\beta$$
$$z = \alpha^2 + \beta^2$$

Choice 8: Equation (11.3) can be written in ratio form as

$\frac{x-7z}{9z-y} = \frac{9z+y}{x+7z} = \frac{\alpha}{\beta}$, $\beta \neq 0$

This equation is equal to the following two equations:

$$\beta x + \alpha y - (7\beta + 9\alpha)z = 0$$

$$-\alpha x + \beta y + (9\beta - 7\alpha)z = 0$$

By the method of cross multiplication, we get the integral solutions of (11.1) to be

$$x = -7\alpha^2 + 7\beta^2 + 18\alpha\beta$$
$$y = 9\alpha^2 - 9\beta^2 + 14\alpha\beta$$
$$z = \alpha^2 + \beta^2$$

PATTERN-3

Assume

$z = z(a,b) = a^2 + b^2$ (11.4)

$$where\ a,b \neq 0,$$

Write

$$130 = 11^2 + 3^2 = (11+3i)(11-3i) \tag{11.5}$$

Substituting (11.4) and (11.5) in (11.1) and employing factorization. Consider

$$x + iy = (11+3i)(a+ib)^2$$

Equating the real and imaginary parts in the above equation, we acquire the integer solution to (11.1) as

$$x = 11a^2 - 11b^2 - 6ab$$
$$y = 3a^2 - 3b^2 + 22ab$$
$$z = a^2 + b^2$$

Also, we can write

$$130 = 3^2 + 11^2 = (3+11i)(3-11i) \tag{11.6}$$

Substituting (11.4) and (11.6) in (11.1) and employing the development of factorization. Write

$$x + iy = (3+11i)(a+ib)^2$$

Equating the real and imaginary parts in the top of the equation, we acquire

$$x = 3a^2 - 3b^2 - 22ab$$
$$y = 11a^2 - 11b^2 + 6ab$$
$$z = a^2 + b^2$$

PATTERN-4

Assume

$$z = z(a,b) = a^2 + b^2$$

$$where a, b \neq 0,$$

Write

$$130 = 7^2 + 9^2 = (7+9i)(7-9i) \tag{11.7}$$

Substituting (11.4) and (11.7) in (11.1) and employing the development of factorization. Write

$$x + iy = (7 + 9i)(a + ib)^2$$

Equating the real and imaginary parts in the top of the equation, we acquire

$$x = 7a^2 - 7b^2 - 18ab$$
$$y = 9a^2 - 9b^2 + 14ab$$
$$z = a^2 + b^2$$

We can also write

$$130 = 9^2 + 7^2 = (9 + 7i)(9 - 7i) \qquad (11.8)$$

Substituting (11.4) and (11.8) in (11.1) and employing the development of factorization.

write

$$x + iy = (9 + 7i)(a + ib)^2$$

Equating the real and imaginary parts in the top of the equation, we acquire

$$x = 9a^2 - 9b^2 - 14ab$$
$$y = 7a^2 - b^2 + 18ab$$
$$z = a^2 + b^2$$

PATTERN-5

Equation (11.1) can be written as

$$x^2 = 130z^2 - y^2 = \left(\sqrt{130}z + y\right)\left(\sqrt{130}z - y\right) \qquad (11.9)$$

Assume

$$x^2 = 130a^2 - b^2 \qquad (11.10)$$

$$wherea, b \neq 0,$$

Using (11.10) in (11.9) and applying method, we have

$$\sqrt{130}z + y = \left(\sqrt{130}a + b\right)^2 = 130a^2 + b^2 + 2\sqrt{130}ab$$

Equating the rational and irrational factors, we get the integer solution to (11.1) as

$$x = 130a^2 - b^2$$
$$y = 130a^2 + b^2$$
$$z = 2ab$$

PATTERN-6

Consider (11.1) as

$$x^2 + y^2 = 130z^2 * 1 \qquad (11.11)$$

$$(x + iy)(x - iy) = (11 + 3i)(11 - 3i)(a + ib)^2(a - ib)^2 * 1$$

Now, $x^2 + y^2 = z^2$

$$\Rightarrow \frac{(x+iy)(x-iy)}{z^2} = 1 \qquad (11.12)$$

Let us take

$$x = 2mn$$
$$y = m^2 - n^2$$
$$z = m^2 + n^2$$

Equation (11.12) can be written as

$$\frac{\left(2mn+i\left(m^2-n^2\right)\right)\left(2mn-i\left(m^2-n^2\right)\right)}{m^2+n^2} = 1 \qquad (11.13)$$

Substituting (11.13) in (11.11) and employing the technique of factorization

write

$$x + iy = (11 + 3i)(a + ib)^2 \frac{\left(2mn+i\left(m^2-n^2\right)\right)}{m^2+n^2} \qquad (11.14)$$

We have that

$$(a + ib)^2 = a^2 - b^2 + i2ab \qquad (11.15)$$

Consider

$$(a + ib)^2 = f(a, b) + ig(a, b)$$

We have

$$f(a,b) = a^2 - b^2$$
$$g(a,b) = 2ab$$

Also, but

$$F(m,n) = 2mn$$
$$G(m,n) = m^2 - n^2$$

Substituting (11.15) in (11.14), we have

$$x + iy = \frac{1}{m^2 + n^2}[(13 + 4i)(f(a,b) + ig(a,b))(F(m,n) + i(G(m,n))]$$

$$x + iy = \frac{1}{m^2 + n^2}[13(fF - gG) + i13(gF + fG) + 4i(fF - gG) - 4(gF - fG)]$$

For integer solutions replace a by$(m^2 + n^2)P$and b by $(m^2 + n^2)Q$

Then, equating the real and imaginary parts , we get the integer solution to (11.1) as

$$x = m^2 + n^2[11(f(P,Q)F - g(P,Q)G) - 3(g(P,Q)F + f(P,Q)G)]$$
$$y = m^2 + n^2[11(g(P,Q)F - f(P,Q)G) + 3(f(P,Q)F - g(P,Q)G)]$$
$$z = (m^2 + n^2)^2(P^2 + Q^2)$$

PATTERN-8

Equation (11.11) can also be written as

$$(x + iy)(x - iy) = (3 + 11i)(3 - 11i)(a + ib)^2(a - ib)^2 * 1 \qquad (11.16)$$

Consider

$$x + iy = (3 + 11i)(a + ib)^2 \frac{\left(2mn + i(m^2 - n^2)\right)}{m^2 + n^2} \qquad (11.17)$$

By using (11.15)

$$\Rightarrow x+iy=\frac{1}{m^2+n^2}\Big[(3+11i)\big(f(a,b)+ig(a,b)\big)\Big(F(m,n) + i\big(G(m,n)\big)\Big)\Big]$$

$$\Rightarrow x+iy=\frac{1}{m^2+n^2}[3(fF-gG)+i3(gF+fG)+11i(fF-gG) - 11(gF+fG)]$$

For integer solutions replace a by $(m^2+n^2)P$and b by $(m^2+n^2)Q$

Then, equating the real and imaginary parts, we get the integer solution to (11.1) as

$$x=m^2+n^2[3(f(P,Q)F-g(P,Q)G)-11(g(P,Q)F+f(P,Q)G)]$$
$$y=m^2+n^2[11(g(P,Q)F-f(P,Q)G)+3(f(P,Q)F-g(P,Q)G)]$$
$$z=(m^2+n^2)^2(P^2+Q^2)$$

PATTERN-9

Equation (11.11) can also be written as

$$(x+iy)(x-iy)=(7+9i)(7-9i)(a+ib)^2(a-ib)^2+1 \qquad (11.18)$$

consider

$$x+iy=(7+9i)(a+ib)^2\frac{\big(2mn+i(m^2-n^2)\big)}{m^2+n^2} \qquad (11.19)$$

$$\Rightarrow x+iy=\frac{1}{m^2+n^2}[(7+9i)(f(a,b)+ig(a,b))(F(m,n) + i(G(m,n))]$$

$$\Rightarrow x+iy=\frac{1}{m^2+n^2}[7(fF-gG)+i7(gf+fg)+9i(fF-gG) - 9(gF+fG)]$$

For integer solution replace a by $(m^2+n^2)P$and b by $(m^2+n^2)Q$

Then, Equating the real and imaginary parts , we get

$$x=m^2+n^2[7(f(P,Q)F-g(P,Q)G)-9(g(P,Q)F+f(P,Q)G)]$$
$$y=m^2+n^2[9(g(P,Q)F-f(P,Q)G)+7(f(P,Q)F-g(P,Q)G)]$$
$$z=(m^2+n^2)^2(P^2+Q^2)$$

PATTERN-10

Equation (11.11) can also be written as

$$(x+iy)(x-iy)=(9+7i)(9-7i)(a+ib)^2(a-ib)^2+1 \tag{11.20}$$

consider

$$x+iy=(9+7i)(a+ib)^2\frac{\left(2mn+i\left(m^2-n^2\right)\right)}{m^2+n^2} \tag{11.21}$$

$$\Rightarrow x+iy=\frac{1}{m^2+n^2}[(9+7i)(f(a,b)+ig(a,b))(F(m,n)+i(G(m,n))]$$

$$\Rightarrow x+iy=\frac{1}{m^2+n^2}[9(fF-gG)+i9(gF+fG)+7i(fF-gG)-7i(gF+fG)]$$

For integer solution replace a by $(m^2+n^2)P$and b by $(m^2+n^2)Q$

Then, Equating the real and imaginary parts , we get

$$x=m^2+n^2[9(f(P,Q)F-g(P,Q)G)-7(g(P,Q)F+f(P,Q)G)]$$
$$y=m^2+n^2[7(g(P,Q)F-f(P,Q)G)+9(f(P,Q)F-g(P,Q)G)]$$
$$z=(m^2+n^2)^2(P^2+Q^2)$$

11.3 Conclusion:

The ternary quadratic Diophantine equations are prosperous in diversity. One possibly will search for further choices of Diophantine equations to discover their consequent integer solutions.

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Chapter 12

Designs of integer solutions to homogeneous ternary quadratic equation $x^2 + y^2 = 370z^2$

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Abstract: The Quadratic Diophantine equation with three unknowns represented by $x^2 + y^2 = 370z^2$ is analyzed for finding its non-zero distinct integral solutions. Different patterns of solutions of the equation under consideration are obtained through factorization technique.

Keywords: Ternary quadratic equation, homogenous quadratic equation, integral solutions, factorization method, ratio form

12.1 Introduction

The Quadratic Diophantine equation with three unknowns offers an unlimited field for research because of their variety (Carmichael., 1959; Dickson., 2005; Mordell., 1970). In particular, one may refer (Gopalan et.al., 2015;Vidhyalakshmi et.al,. 2021 ; Shanthi et.al .,2014,2023) for quadratic equations with three unknowns. This communication concerns with yet another interesting equation $x^2 + y^2 = 370z^2$ representing homogeneous Diophantine equation with three unknowns for determining its infinitely many non-zero integral solutions. A few interesting properties among its solutions are given.

12.2Method of analysis

The ternary quadratic Diophantine equation to solved for its non-zero distinct integral solution is

$x^2 + y^2 = 370z^2$ (12.1)

Different patterns of solution of (12.1) are presented below.

PATTERN -1

Equation (12.1) can be written as

$$x^2 + y^2 = 361z^2 + 9z^2$$

$(x + 19z)(x - 19z) = (3z + y)(3z - y)$ (12.2)

The process of solving (12.2) is illustrated as below:

Choice 1: Equation (12.2) can be written in ratio form as

$\frac{x+19z}{3z+y} = \frac{3z-y}{x-19z} = \frac{\alpha}{\beta}$, $\beta \neq 0$

This equation is equal to the following two equations:

$$\beta x - \alpha y + (19\beta - 3\alpha)z = 0$$

$$-\alpha x - \beta y + (3\beta + 19\alpha)z = 0$$

By the method of cross multiplication, we get the integral solutions of (12.1) to be

$$x = 19\alpha^2 - 19\beta^2 + 6\alpha\beta$$
$$y = -3\alpha^2 + 3\beta^2 + 38\alpha\beta$$
$$z = \alpha^2 + \beta^2$$

Choice 2: Equation (12.2) can be written in ratio form as

$\frac{x+19z}{3z-y} = \frac{3z+y}{x-19z} = \frac{\alpha}{\beta}$, $\beta \neq 0$

This equation is equal to the following two equations:

$$\beta x + \alpha y + (19\beta - 3\alpha)z = 0$$

$$-\alpha x + \beta y + (19\alpha + 3\beta)z = 0$$

By the method of cross multiplication, we get the integral solutions of (12.1) to be

$$x = 19\alpha^2 - 19\beta^2 + 6\alpha\beta$$
$$y = 3\alpha^2 - 3\beta^2 - 38\alpha\beta$$
$$z = \alpha^2 + \beta^2$$

Choice 3: Equation (12.2) can be written in ratio as

$$\frac{x-19z}{3z+y} = \frac{3z-y}{x+19z} = \frac{\alpha}{\beta} \quad ,\beta \neq 0$$

This equation is equal to the following two equations:

$$\beta x - \alpha y - (19\beta + 3\alpha)z = 0$$

$$-\alpha x - \beta y + (3\beta - 19\alpha)z = 0$$

By the method of cross multiplication, we get the integral solutions of (12.1) to be

$$x = 19\alpha^2 - 19\beta^2 - 6\alpha\beta$$
$$y = 3\alpha^2 - 3\beta^2 + 38\alpha\beta$$
$$z = -\alpha^2 - \beta^2$$

Choice 4: Equation (12.2) can be written in ratio form as

$$\frac{x-19z}{3z-y} = \frac{3z+y}{x+19z} = \frac{\alpha}{\beta} \quad ,\beta \neq 0$$

This equation is equal to the following two equations:

$$\beta x + \alpha y - (19\beta + 3\alpha)z = 0$$

$$-\alpha x + \beta y + (3\beta - 19\alpha)z = 0$$

By the method of cross multiplication, we get the integral solutions of (12.1) to be

$$x = -19\alpha^2 + 19\beta^2 + 6\alpha\beta$$

$$y = 3\alpha^2 - 3\beta^2 + 38\alpha\beta$$
$$z = \alpha^2 + \beta^2$$

PATTERN-2

Equation (12.1) can be written as

$$x^2 + y^2 = 289z^2 + 81z^2$$

$$(x + 17z)(x - 17z) = (9z + y)(9z - y) \tag{12.3}$$

The process of solving (12.3) is illustrated as below:

Choice 5: Equation (12.3) can be written in ratio form as

$$\frac{x+17z}{9z+y} = \frac{9z-y}{x-17z} = \frac{\alpha}{\beta}, \quad \beta \neq 0$$

This equation is equal to the following two equations:

$$\beta x - \alpha y + (17\beta - 9\alpha)z = 0$$

$$-\alpha x - \beta y + (9\beta + 17\alpha)z = 0$$

By the method of cross multiplication, we get the integral solutions of (12.1) to be

$$x = -17\alpha^2 + 17\beta^2 - 18\alpha\beta$$
$$y = 9\alpha^2 - 9\beta^2 - 34\alpha\beta$$
$$z = -\alpha^2 - \beta^2$$

Choice 6: Equation (12.3) can be written in ratio form as

$$\frac{x+17z}{9z-y} = \frac{9z+y}{x-17z} = \frac{\alpha}{\beta}, \quad \beta \neq 0$$

This equation is equal to the following two equations:

$$\beta x + \alpha y + (17\beta - 9\alpha)z = 0$$

$$-\alpha x + \beta y + (17\alpha + 9\beta)z = 0$$

By the method of cross multiplication, we get the integral solutions of (12.1) to be

$$x = 17\alpha^2 - 17\beta^2 + 18\alpha\beta$$
$$y = 9\alpha^2 - 9\beta^2 - 34\alpha\beta$$

$$z = \alpha^2 + \beta^2$$

Choice 7: Equation (12.3) can be written in ratio form as

$$\frac{x-17z}{9z+y} = \frac{9z-y}{x+17z} = \frac{\alpha}{\beta} \;, \qquad \beta \neq 0$$

This equation is equal to the following two equations:

$$\beta x - \alpha y - (17\beta + 9\alpha)z = 0$$

$$-\alpha x - \beta y + (9\beta - 17\alpha)z = 0$$

By the method of cross multiplication, we get the integral solutions of (12.1) to be

$$x = 17\alpha^2 - 17\beta^2 - 18\alpha\beta$$
$$y = 9\alpha^2 - 9\beta^2 + 34\alpha\beta$$
$$z = -\alpha^2 - \beta^2$$

Choice 8: Equation (12.3) can be written in ratio form as

$$\frac{x-17z}{9z-y} = \frac{9z+y}{x+17z} = \frac{\alpha}{\beta} \;, \qquad \beta \neq 0$$

This equation is equal to the following two equations:

$$\beta x + \alpha - (17\beta + 9\alpha)z = 0$$

$$-\alpha x + \beta y - (9\beta - 17\alpha)z = 0$$

By the method of cross multiplication, we get the integral solutions of (12.1) to be

$$x = -17\alpha^2 + 17\beta^2 + 18\alpha\beta$$
$$y = 9\alpha^2 - 9\beta^2 + 34\alpha\beta$$
$$z = \alpha^2 + \beta^2$$

PATTERN-3

Assume

$$z = z(a,b) = a^2 + b^2 \tag{12.4}$$

$$where\ a, b \neq 0,$$

Write

$$370 = 19^2 + 3^2 = (19 + 3i)(19 - 3i) \tag{12.5}$$

Substituting (12.4) and (12.5) in (12.1) and employing factorization. Consider

$$x + iy = (19 + 3i)(a + ib)^2$$

Equating the real and imaginary parts in the above equation, we acquire the integer solution to (12.1) as

$$x = 19a^2 - 19b^2 - 6ab$$
$$y = 3a^2 - 3b^2 + 38ab$$
$$z = a^2 + b^2$$

Also, we can write

$$370 = 3^2 + 19^2 = (3 + 19i)(3 - 19i) \tag{12.6}$$

Substituting (12.4) and (12.6) in (12.1) and employing the development of factorization.

write

$$x + iy = (3 + 19i)(a + ib)^2$$

Equating the real and imaginary parts in the top of the equation, we acquire

$$x = 3a^2 - 3b^2 - 38ab$$
$$y = 19a^2 - 19b^2 + 6ab$$
$$z = a^2 + b^2$$

PATTERN-4

Assume

$$z = z(a, b) = a^2 + b^2$$

$$where a, b \neq 0,$$

Write

$$370 = 17^2 + 9^2 = (17 + 9i)(17 - 9i) \tag{12.7}$$

Substituting (12.4) and (12.7) in (12.1) and employing the development of factorization.

write

$$x + iy = (17 + 9i)(a + ib)^2$$

Equating the real and imaginary parts in the top of the equation, we acquire

$$x = 17a^2 - 17b^2 - 18ab$$
$$y = 9a^2 - 9b^2 + 34ab$$
$$z = a^2 + b^2$$

We can also write

$$370 = 9^2 + 17^2 = (9 + 17i)(9 - 17i) \qquad (12.8)$$

Substituting (12.4) and (12.8) in (12.1) and employing the development of factorization.

Write

$$x + iy = (9 + 17i)(a + ib)^2$$

Equating the real and imaginary parts in the top of the equation, we acquire

$$x = 9a^2 - 9b^2 - 34ab$$
$$y = 17a^2 - 17b^2 + 18ab$$
$$z = a^2 + b^2$$

PATTERN-5

Equation (12.1) can be written as

$$x^2 = 370z^2 - y^2 = \left(\sqrt{370}z + y\right)\left(\sqrt{370}z - y\right) \qquad (12.9)$$

Assume

$$x^2 = 370a^2 - b^2 \qquad (12.10)$$

$$where a, b \neq 0,$$

Using (12.10) in (12.9) and applying method, we have

$$\sqrt{370}z + y = \left(\sqrt{370}a + b\right)^2$$

Equating the rational and irrational factors, we get the integer solution to (12.1) as

$$x = 370a^2 - b^2$$
$$y = 370a^2 + b^2$$
$$z = 2ab$$

PATTERN-6

Consider (12.1) as

$$x^2 + y^2 = 370z^2 * 1 \qquad (12.11)$$

$$(x + iy)(x - iy) = (19 + 3i)(19 - 3i)(a + ib)^2(a - ib)^2 * 1$$

Now, $x^2 + y^2 = z^2$

$$\Rightarrow \frac{(x+iy)(x-iy)}{z^2} = 1 \qquad (12.12)$$

Let us take

$$x = 2mn$$
$$y = m^2 - n^2$$
$$z = m^2 + n^2$$

Equation (12.12) can be written as

$$\frac{\left(2mn+i\left(m^2-n^2\right)\right)\left(2mn-i\left(m^2-n^2\right)\right)}{m^2+n^2} = 1 \qquad (12.13)$$

Substituting (12.13) in (12.11) and employing the technique of factorization,

write

$$x + iy = (19 + 3i)(a + ib)^2 \frac{\left(2mn+i\left(m^2-n^2\right)\right)}{m^2+n^2} \qquad (12.14)$$

We have that

$$(a+ib)^2 = a^2 - b^2 + i2ab \qquad (12.15)$$

Consider

$$(a+ib)^2 = f(a,b) + ig(a,b)$$

We have

$$f(a,b) = a^2 - b^2$$
$$g(a,b) = 2ab$$

Also, but

$$F(m,n) = 2mn$$
$$G(m,n) = m^2 - n^2$$

Substituting (12.15) in (12.14), we have

$$x + iy = \frac{1}{m^2+n^2}[(19+3i)(f(a,b)+ig(a,b))(F(m,n)+i(G(m,n))]$$

$$x + iy = \frac{1}{m^2+n^2}[19(fF - gG) + i19(gF + fG) + 3i(fF - gG) - 3(gF - fG)]$$

For integer solutions replace a by$(m^2+n^2)P$and b by $(m^2+n^2)Q$

Then, equating the real and imaginary parts , we get the integer solution to (12.1) as

$$x = m^2 + n^2[19(f(P,Q)F - g(P,Q)G) - 3(g(P,Q)F + f(P,Q)G)]$$
$$y = m^2 + n^2[19(g(P,Q)F - f(P,Q)G) + 3(f(P,Q)F - g(P,Q)G)]$$
$$z = (m^2+n^2)^2(P^2+Q^2)$$

PATTERN-8

Equation (12.11) can also be written as

$$(x+iy)(x-iy) = (3+19i)(3-19i)(a+ib)^2(a-ib)^2 * 1 \qquad (12.16)$$

Consider

$$x + iy = (3+19i)(a+ib)^2 \frac{\left(2mn + i(m^2-n^2)\right)}{m^2+n^2} \qquad (12.17)$$

By using (12.15)

$$\Rightarrow x + iy = \frac{1}{m^2+n^2}\left[(3+19i)\big(f(a,b)+ig(a,b)\big)\Big(F(m,n) + i\big(G(m,n)\big)\Big)\right]$$

$$\Rightarrow x + iy = \frac{1}{m^2+n^2}[3(fF-gG)+i3(gF+fG)+19i(fF-gG) - 19(gF+fG)]$$

For integer solutions replace a by $(m^2+n^2)P$and b by $(m^2+n^2)Q$

Then, equating the real and imaginary parts, we get the integer solution to (12.1) as

$$x = m^2+n^2[3(f(P,Q)F-g(P,Q)G)-19(g(P,Q)F+f(P,Q)G)]$$
$$y = m^2+n^2[19(g(P,Q)F-f(P,Q)G)+3(f(P,Q)F-g(P,Q)G)]$$
$$z = (m^2+n^2)^2(P^2+Q^2)$$

PATTERN-9

Equation (12.11) can also be written as

$$(x+iy)(x-iy) = (17+9i)(17-9i)(a+ib)^2(a-ib)^2+1 \quad (12.18)$$

consider

$$x+iy = (17+9i)(a+ib)^2\frac{\big(2mn+i(m^2-n^2)\big)}{m^2+n^2} \quad (12.19)$$

$$\Rightarrow x + iy = \frac{1}{m^2+n^2}[(17+9i)(f(a,b)+ig(a,b))(F(m,n) + i(G(m,n))]$$

$$\Rightarrow x + iy = \frac{1}{m^2+n^2}[17(fF-gG)+i17(gf+fg)+9i(fF-gG) - 9(gF+fG)]$$

For integer solution replace a by $(m^2+n^2)P$and b by $(m^2+n^2)Q$

Then, Equating the real and imaginary parts , we get

$$x = m^2 + n^2[17(f(P,Q)F - g(P,Q)G) - 9(g(P,Q)F + f(P,Q)G)]$$
$$y = m^2 + n^2[9(g(P,Q)F - f(P,Q)G) + 17(f(P,Q)F - g(P,Q)G)]$$
$$z = (m^2 + n^2)^2(P^2 + Q^2)$$

PATTERN-10

Equation (12.11) can also be written as

$$(x + iy)(x - iy) = (9 + 17i)(9 - 17i)(a + ib)^2(a - ib)^2 + 1 \qquad (12.20)$$

consider

$$x + iy = (9 + 17i)(a + ib)^2 \frac{\left(2mn + i(m^2 - n^2)\right)}{m^2 + n^2} \qquad (12.21)$$

$$\Rightarrow x + iy = \frac{1}{m^2 + n^2}[(9 + 17i)(f(a,b) + ig(a,b))(F(m,n) + i(G(m,n))]$$

$$\Rightarrow x + iy = \frac{1}{m^2 + n^2}[9(fF - gG) + i9(gF + fG) + 17i(fF - gG) - 17i(gF + fG)]$$

For integer solution replace a by $(m^2 + n^2)P$and b by $(m^2 + n^2)Q$

Then, Equating the real and imaginary parts , we get

$$x = m^2 + n^2[9(f(P,Q)F - g(P,Q)G) - 17(g(P,Q)F + f(P,Q)G)]$$
$$y = m^2 + n^2[9(g(P,Q)F - f(P,Q)G) + 17(f(P,Q)F - g(P,Q)G)]$$
$$z = (m^2 + n^2)^2(P^2 + Q^2)$$

12.3 Conclusion:

The ternary quadratic Diophantine equations are prosperous in diversity. One possibly will search for further choices of Diophantine equations to discover their consequent integer solutions.

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