DeepScience

An Ensemble of Multivariable Higher degree Diophantine and Transcendental Equations

M.A.Gopalan
N. Thiruniraiselvi
S. Devibala
Sharadha Kumar

An Ensemble of Multivariable Higher degree Diophantine and Transcendental Equations

M. A. Gopalan

Department of Mathematics, Shrimati Indira Gandhi College, Affiliated to Bharathidasan University, Trichy-620 002,Tamil Nadu, India

N. Thiruniraiselvi

Department of Mathematics, Shrimati Indira Gandhi College, Affiliated to Bharathidasan University, Trichy-620 002,Tamil Nadu, India

S. Devibala

Mathematics,Sri Meenakshi Govt Arts College for Women(A), Madurai-625002, Tamil Nadu, India

Sharadha Kumar

Department of Mathematics, National College, Affiliated to Bharathidasan University, Trichy- 620 001, Tamil Nadu, India

DeepScience

Published, marketed, and distributed by:

Deep Science Publishing
USA | UK | India | Turkey
Reg. No. MH-33-0523625
www.deepscienceresearch.com
editor@deepscienceresearch.com
WhatsApp: +91 7977171947

ISBN: 978-93-49307-06-3

E-ISBN: 978-93-49307-38-4

https://doi.org/10.70593/978-93-49307-38-4

Citation: Gopalan, M. A., Thiruniraiselvi, N., Devibala, S., & Kumar, S. (2025). *An Ensemble of Multivariable Higher degree Diophantine and Transcendental Equations*. Deep Science Publishing. https://doi.org/10.70593/978-93-49307-38-4

Preface

A vast and fascinating field of mathematics in number theory is the subject of Diophantine equations consisting of the study of polynomial equations usually involving two or more parameters such that only solutions in integers are concentrated. The mathematical study of Diophantine problems that Diophantus initiated is Diophantine analysis. Diophantine problems have fewer equations than unknown variables and involve finding integers that work correctly for all equations.

In studies of Diophantine equations of degrees higher than two, significant success was attained only in the 20th century. There has been interest in determining all integer solutions to multi variables and higher degree Diophantine equations among mathematicians. In this context, for simplicity and brevity, one may refer (Carmichael.,1959, Dickson.,1952, Mordell.,1969, Gopalan et.al., 2012a, Gopalan et.al., 2015b, Gopalan et.al., 2024c, Mahalakshmi, Shanthi.,2023a, Mahalakshmi, Shanthi.,2023b, Mahalakshmi, Shanthi.,2023c, Sathiyapriya et.al., 2024a, Sathiyapriya et.al., 2024b, Shanthi.,2023a, Shanthi.,2023b, Shanthi, Mahalakshmi.,2023a, Shanthi, Mahalakshmi.,2023b, Shanthi, Mahalakshmi.,2023c, Shanthi, Gopalan.,2024a, Shanthi, Gopalan.,2024b, Thiruniraiselvi, Gopalan., 2024a, Vidhyalakshmi et.al., 2022a) for some binary and ternary quadratic Diophantine equations.

Note that, the non-algebraic equations can be solved by transforming it into an equivalent polynomial equation. Some transcendental equation in more than one unknown can be solved by separation of the unknowns reducing them to polynomial equations (Thiruniraiselvi, Gopalan., 2024b; Vidhyalakshmi et.al., 2021b).

The focus in this book is on solving multivariable higher degree Diophantine equations along with transcendental equations. These types of equations are significant since they concentrate on obtaining solutions in integers which satisfy the considered algebraic and transcendental equations. These solutions play a vital role in different area of mathematics & science and help us in understanding the significance of number patterns. This book contains a reasonable collection of special multivariable higher degree Diophantine problems & transcendental equations with three and five unknowns. The procedure in obtaining varieties of solutions in integers for the polynomial and transcendental Diophantine equations considered in this book are illustrated in an elegant manner.

Dr. M. A. Gopalan,
Dr. N. Thiruniraiselvi,
Dr. S. Devibala,
Dr. Sharadha Kumar.

Contents

CHAPTER	TITLE	PAGE NUMBER
1.	A PEER SEARCH ON BINARY QUADRATIC DIOPHANTINE EQUATION	1
2.	ON FINDING INTEGER SOLUTIONS TO HOMOGENEOUS TERNARY QUADRATIC DIOPHANTINE EQUATION	10
3.	A PEER SEARCH ON QUATERNARY QUADRATIC DIOPHANTINE EQUATION	14
4.	ON NON-HOMOGENEOUS TERNARY CUBIC EQUATION	20
5.	A PEER SEARCH ON TERNARY CUBIC DIOPHANTINE EQUATION	23
6.	ON NON-HOMOGENEOUS QUATERNARY QUARTIC EQUATION	29
7.	A PEER SEARCH ON QUINARY CUBIC EQUATION	38
8.	TECHNIQUES TO SOLVE NON-HOMOGENEOUS QUATERNARY QUARTIC DIOPHANTINE EQUATION	45
9.	ON QUATERNARY SEXTIC EQUATION	48
10.	ON NON-HOMOGENEOUS TERNARY HEPTIC DIOPHANTINE EQUATION	52
11.	ON NON-HOMOGENEOUS QUINARY HEPTIC EQUATION	62
12.	ON THE TERNARY SURD EQUATION	68
13.	ON THE QUINARY SURD EQUATION	72
	CONCLUSION	77
	REFERENCES	78

Notations

Figurate numbers	Notation for rank n	Definitions
Gnomonic Number	Gno_n	$2n-1$
Pentagonal pyramidal Number	PP_n	$\frac{1}{2}n^2(n+1)$
Star Number	S_n	$6n(n-1)+1$
Regular Polygonal Number	$t_{m,n}$	$n\left(1+\frac{(n-1)(m-2)}{2}\right)$
Pronic Number	Pr_n	$n(n+1)$
Pyramidal Number	P_n^m	$\frac{n(n+1)}{6}[(m-2)n+(5-m)]$
Centered Pyramidal Number	$CP_{m,n}$	$\frac{m(n-1)n(n+1)}{6}+n$
Stella octangular Number	SO_n	$n(2n^2-1)$

Deep Science Publishing
https://doi.org/10.70593/978-93-49307-38-4

Chapter 1

A PEER SEARCH ON BINARY QUADRATIC DIOPHANTINE EQUATION

1.1 Technical Procedure

The non-homogeneous binary quadratic equation for finding its integer solutions is represented by

$$x^2 + bxy + cy^2 = 1 \quad (1.1)$$

where b, c are any given non-zero integers such that

$$b^2 - 4c > 0 \text{ and square-free} \quad (1.2)$$

The technical process of solving (1.1) is illustrated below:

Process 1

Treating (1.1) as a quadratic in x and solving for the same , we have the pellian equation

$$X^2 = Dy^2 + 4 \quad (1.3)$$

where

$$D = b^2 - 4c$$

and

$$X = 2x + by \quad (1.4)$$

The smallest positive integer solutions to (1.3) are denoted by

$$y = y_0 ,\quad X = X_0 .$$

To obtain the other solutions to (1.3) , consider the equation

$$X^2 = D y^2 + 1 \tag{1.5}$$

whose general solutions are denoted by

$$y = \tilde{y}_n = \frac{1}{2\sqrt{D}} g_n ,$$
$$X = \tilde{X}_n = \frac{1}{2} f_n$$

Where

$$f_n = (\tilde{X}_0 + \sqrt{D}\,\tilde{y}_0)^{n+1} + (\tilde{X}_0 - \sqrt{D}\,\tilde{y}_0)^{n+1} ,$$
$$g_n = (\tilde{X}_0 + \sqrt{D}\,\tilde{y}_0)^{n+1} - (\tilde{X}_0 - \sqrt{D}\,\tilde{y}_0)^{n+1}$$

in which $\tilde{y}_0 , \tilde{X}_0$ represent the least positive integer solutions to (1.5).

Employing the lemma of Brahmagupta between the solutions $(y_0 , X_0) \& (\tilde{y}_n , \tilde{X}_n)$, the other solutions to (1.3) are given by

$$y_{n+1} = y_0 \tilde{X}_n + X_0 \tilde{y}_n = \frac{1}{2} y_0 f_n + \frac{1}{2\sqrt{D}} X_0 g_n \tag{1.6}$$

and

$$X_{n+1} = X_0 \tilde{X}_n + D y_0 \tilde{y}_n = \frac{1}{2} X_0 f_n + \frac{1}{2} y_0 \sqrt{D}\, g_n \tag{1.7}$$

In view of (1.4) ,we have from (1.6) & (1.7)

$$\begin{aligned} x_{n+1} &= \frac{1}{2}[X_{n+1} - b y_{n+1}] \\ &= \frac{1}{4}(X_0 - b y_0) f_n + \frac{1}{4\sqrt{D}}(D y_0 - b X_0) g_n \end{aligned} \tag{1.8}$$

Thus, (1.6) & (1.8) satisfy (1.1) .

The recurrence relations satisfied by the values of $x_{n+1}, y_{n=1}$ are given below:

$$x_{n+3} - 2\,\tilde{X}_0\, x_{n+2} + x_{n+1} = 0,$$
$$y_{n+3} - 2\,\tilde{X}_0\, y_{n+2} + y_{n+1} = 0, n = -1,0,1,... \qquad (1.9)$$

To analyze the nature of solutions, one has to go in for solutions of (1.1) when $b\,\&\,c$ take particular values. A few examples are exhibited below:

Example 1

Let $b = 9, c = 15$ .Note that (1.2) is satisfied and $D = 21$.

Further , observe that

$$y_0 = 1\ , X_0 = 5$$
$$\tilde{y}_0 = 12, \tilde{X}_0 = 55$$

It is seen that the general solutions to

$$x^2 + 9xy + 15y^2 = 1 \qquad (1.10)$$

from (1.6) & (1.8) are given by

$$x_{n+1} = -f_n - \frac{6}{\sqrt{21}} g_n\ ,$$
$$y_{n+1} = \frac{1}{2} f_n + \frac{5}{2\sqrt{21}} g_n$$

where

$$f_n = (55 + 12\sqrt{21})^{n+1} + (55 - 12\sqrt{21})^{n+1}\ ,$$
$$g_n = (55 + 12\sqrt{21})^{n+1} - (55 - 12\sqrt{21})^{n+1}\ .$$

A few numerical solutions to (1.10) are presented below:

$$x_0 = -2\ , y_0 = 1$$
$$x_1 = -254, y_1 = 115$$
$$x_2 = -27938, y_2 = 12649$$

The recurrence relations satisfied by the solutions to (1.10) are presented below:

$$x_{n+3} - 110x_{n+2} + x_{n+1} = 0 \ ,$$
$$y_{n+3} - 110y_{n+2} + y_{n+1} = 0 , n = -1,0,1,...$$

Example 2

Let $b = 1, c = -1$ .Note that (1.2) is satisfied and $D = 5$.

Further , observe that

$$y_0 = 1 \ , X_0 = 3$$
$$\tilde{y}_0 = 4 , \tilde{X}_0 = 9$$

It is seen that the general solutions to

$$x^2 + x\,y - y^2 = 1 \qquad (1.11)$$

From (1.6) & (1.8) are given by

$$x_{n+1} = \frac{1}{2} f_n + \frac{1}{2\sqrt{5}} g_n \ ,$$
$$y_{n+1} = \frac{1}{2} f_n + \frac{3}{2\sqrt{5}} g_n$$

where

$$f_n = (9 + 4\sqrt{5})^{n+1} + (9 - 4\sqrt{5})^{n+1} \ ,$$
$$g_n = (9 + 4\sqrt{5})^{n+1} - (9 - 4\sqrt{5})^{n+1} \ .$$

A few numerical solutions to (11) are presented below:

$$x_0 = 1 \ , y_0 = 1$$
$$x_1 = 13, y_1 = 21$$
$$x_2 = 233, y_2 = 377$$

The recurrence relations satisfied by the solutions to (1.11) are presented below:

$$x_{n+3} - 18x_{n+2} + x_{n+1} = 0 \ ,$$
$$y_{n+3} - 18y_{n+2} + y_{n+1} = 0 , n = -1,0,1,...$$

Note 1

It is worth to observe that the lemma of Brahmagupta presented in (1.6) & (1.7) may also be considered as below:

$$y_{n+1} = y_0 \tilde{X}_n - X_0 \tilde{y}_n ,$$
$$X_{n+1} = X_0 \tilde{X}_n - D y_0 \tilde{y}_n .$$

Following the process presented above, another set of solutions to (1.1) are obtained.

Process 2

Treating (1.1) as a quadratic in y and solving for the same, we have the pellian equation

$$Y^2 = D x^2 + 4c \tag{1.12}$$

where

$$Y = 2\, c\, y + b x \tag{1.13}$$

The smallest positive integer solutions to (1.12) are denoted by

$$x = x_0 , \quad Y = Y_0 .$$

For getting more solutions to (1.12) , write

$$Y^2 = D x^2 + 1 \tag{1.14}$$

Which is satisfied by

$$x = \tilde{x}_n = \frac{1}{2\sqrt{D}} g_n ,$$
$$Y = \tilde{Y}_n = \frac{1}{2} f_n$$

where

$$f_n = (\tilde{Y}_0 + \sqrt{D}\, \tilde{x}_0)^{n+1} + (\tilde{Y}_0 - \sqrt{D}\, \tilde{x}_0)^{n+1} ,$$
$$g_n = (\tilde{Y}_0 + \sqrt{D}\, \tilde{x}_0)^{n+1} - (\tilde{Y}_0 - \sqrt{D}\, \tilde{x}_0)^{n+1}$$

In which $\tilde{Y}_0 , \tilde{x}_0$ represent the least positive integer solutions to (1.14).

Employing the lemma of Brahmagupta between the solutions (x_0 , Y_0) & $(\tilde{x}_n , \tilde{Y}_n)$, the other solutions to (1.12) are given by

$$x_{n+1} = x_0 \tilde{Y}_n + Y_0 \tilde{x}_n = \frac{1}{2} x_0 f_n + \frac{1}{2\sqrt{D}} Y_0 g_n \tag{1.15}$$

and

$$Y_{n+1} = Y_0 \tilde{Y}_n + D x_0 \tilde{x}_n = \frac{1}{2} Y_0 f_n + \frac{1}{2} x_0 \sqrt{D} g_n \tag{1.16}$$

In view of (1.13) ,we have from (1.15) & (1.16)

$$\begin{aligned} y_{n+1} &= \frac{1}{2c}[Y_{n+1} - b x_{n+1}] \\ &= \frac{1}{4c}(Y_0 - b x_0) f_n + \frac{1}{4c\sqrt{D}}(D x_0 - b Y_0) g_n \end{aligned} \tag{1.17}$$

Thus, (1.15) & (1.17) satisfy (1.1) .

The recurrence relations satisfied by the values of $x_{n+1} , y_{n=1}$ are given below:

$$\begin{aligned} & x_{n+3} - 2 \tilde{Y}_0 x_{n+2} + x_{n+1} = 0, \\ & y_{n+3} - 2 \tilde{Y}_0 y_{n+2} + y_{n+1} = 0, n = -1,0,1,... \end{aligned} \tag{1.18}$$

To analyze the nature of solutions, one has to go in for solutions of (1.1) when b & c take particular values. A few examples are exhibited below:

Example 3

Let $b = 9, c = 15$ .Note that (1.2) is satisfied and $D = 21$.

Further, observe that

$$\begin{aligned} & x_0 = 1 , Y_0 = 9 \\ & \tilde{x}_0 = 12, \tilde{Y}_0 = 55 \end{aligned}$$

It is seen that the general solutions to (1.10) from (1.15) & (1.17) are given by

$$x_{n+1} = \frac{1}{2} f_n + \frac{9}{2\sqrt{21}} g_n \,,$$

$$y_{n+1} =_n - \frac{1}{\sqrt{21}} g_n$$

where

$$f_n = (55+12\sqrt{21})^{n+1} + (55-12\sqrt{21})^{n+1} \,,$$
$$g_n = (55+12\sqrt{21})^{n+1} - (55-12\sqrt{21})^{n+1} \,.$$

A few numerical solutions to (1.10) are presented below:

$$x_0 = 1, y_0 = 0$$
$$x_1 = 163, y_1 = -24$$
$$x_2 = 17929, y_2 = -2640$$

The recurrence relations satisfied by the solutions to (1.10) are presented below:

$$x_{n+3} - 110 x_{n+2} + x_{n+1} = 0 \,,$$
$$y_{n+3} - 110 y_{n+2} + y_{n+1} = 0, n = -1,0,1,...$$

Example 4

Let $b = 1, c = -1$ .Note that (1.2) is satisfied and $D = 5$.

Further, observe that

$$x_0 = 1 \,, Y_0 = 1$$
$$\tilde{x}_0 = 4, \tilde{Y}_0 = 9$$

It is seen that the general solutions to (1.11) from (1.15) & (1.17) are given by

$$x_{n+1} = \frac{1}{2} f_n + \frac{1}{2\sqrt{5}} g_n \,,$$

$$y_{n+1} = -\frac{1}{\sqrt{5}} g_n$$

where

$$f_n = (9+4\sqrt{5})^{n+1} + (9-4\sqrt{5})^{n+1},$$
$$g_n = (9+4\sqrt{5})^{n+1} - (9-4\sqrt{5})^{n+1}.$$

A few numerical solutions to (1.11) are presented below:

$$x_0 = 1,\ y_0 = 0$$
$$x_1 = 13,\ y_1 = -8$$
$$x_2 = 233,\ y_2 = -144$$

The recurrence relations satisfied by the solutions to (1.11) are presented below:

$$x_{n+3} - 18x_{n+2} + x_{n+1} = 0,$$
$$y_{n+3} - 18y_{n+2} + y_{n+1} = 0,\ n = -1,0,1,...$$

Process 3

It is worth to remind that, if $x = x_0, y = y_0$ represent the smallest positive integer solutions to the pellian equation $x^2 = Dy^2 + 1$, then $x = \sigma x_0, y = \sigma y_0$ give the integer solutions to the pellian equation $x^2 = Dy^2 + \sigma^2$.

In view of the above result , the general solutions $y = y_n, X = X_n$ to (1.3) are given by

$$X_n + \sqrt{D}\, y_n = 2(\tilde{X}_0 + \sqrt{D}\, \tilde{y}_0)^{n+1},\ n = 0,1,2,... \tag{1.19}$$

where $(\tilde{y}_0, \tilde{X}_0)$ is the smallest positive integer solution of the pellian equation given by (1.5).

Comparing the coefficients of corresponding terms in (19) , we have

$$X_n = (\tilde{X}_0 + \sqrt{D}\, \tilde{y}_0)^{n+1} + (\tilde{X}_0 - \sqrt{D}\, \tilde{y}_0)^{n+1},$$
$$y_n = \frac{1}{\sqrt{D}}[(\tilde{X}_0 + \sqrt{D}\, \tilde{y}_0)^{n+1} - (\tilde{X}_0 - \sqrt{D}\, \tilde{y}_0)^{n+1}]. \tag{1.20}$$

In view of (1.4) , we get

$$x_n = \frac{1}{2}[X_n - b\,y_n]$$
$$= \frac{1}{2}\,[\{\frac{\sqrt{D}-b}{\sqrt{D}}\}(\tilde{X}_0 + \sqrt{D}\,\tilde{y}_0)^{n+1} + \{\frac{\sqrt{D}+b}{\sqrt{D}}\}(\tilde{X}_0 - \sqrt{D}\,\tilde{y}_0)^{n+1}] \tag{1.21}$$

Thus, the values of x_n, y_n given by (1.20) & (1.21) satisfy (1.1).

The values of y_n, x_n satisfy respectively, the following difference equations

$$y_{n+2} - 2\tilde{X}_0 y_{n+1} + y_n = 0,$$
$$x_{n+2} - 2\tilde{X}_0 x_{n+1} + x_n = 0,\ n = 0,1,2,...$$

Deep Science Publishing
https://doi.org/10.70593/978-93-49307-38-4

Chapter 2

On Finding Integer Solutions to Homogeneous Ternary Quadratic Diophantine Equation

2.1 Technical Procedure

The homogeneous second degree equation in three unknowns to be solved is

$$x^2 + (2k+1)\,y^2 = (k+1)^2\,z^2 \tag{2.1}$$

To start with, (2.1) is satisfied by

$$x = 4k^3 + 6k^2 + 3k,\; y = 2k+1,\; z = 4k^2 + 2k + 1.$$

However, there are many more integer solutions and the process of obtaining various solution patterns is illustrated below:

Process 1

Taking

$$x = (k+1)\,X,\; y = (k+1)\,Y \tag{2.2}$$

in (2.1) ,it gives

$$X^2 + (2k+1)\,Y^2 = z^2 \tag{2.3}$$

which is satisfied by

$$Y = 2pq,\; X = (2k+1)\,p^2 - q^2 \tag{2.4}$$

and

$$z = (2k+1)\,p^2 + q^2 \tag{2.5}$$

Using (2.4) in (2.2) ,we have

$$x = (k+1)\,[\,(2k+1)\,p^2 - q^2\,]\,,$$
$$y = 2(k+1)\,p\,q. \qquad (2.6)$$

Thus ,(2.5) & (2.6) represent the integer solutions to (2.1).

Process 2

Consider (2.3) as the system of double equations as shown below

$$\mathrm{z} + \mathrm{X} = \mathrm{Y}^2$$
$$\mathrm{z} - \mathrm{X} = 2\mathrm{k} + 1$$

Solving the above pair of equations, we have

$$Y = 2s + 1\,, X = 2s^2 + 2s - k \qquad (2.7)$$

and

$$z = 2s^2 + 2s + k + 1 \qquad (2.8)$$

From (2.7) and (2.2) ,we get

$$x = (k+1)\;(2s^2 + 2s - k)\;,$$
$$y = (k+1)(2s+1). \qquad (2.9)$$

Thus, (2.8) & (2.9) satisfy (2.1).

Note 1

It is to be noted that, one may write (2.3) as the pair of equations as follows:

$$\mathrm{z} + \mathrm{X} = (2\mathrm{k}+1)\;\mathrm{Y}^2;\; \mathrm{z} - \mathrm{X} = 1$$

In this case, the solutions to (2.1) are obtained as

$$x = (k+1)\,[k\,(2s+1)^2 + 2s^2 + 2s]\,,$$
$$y = (k+1)\;(2s+1)\,,$$
$$z = [k\,(2s+1)^2 + 2s^2 + 2s + 1].$$

Process 3

The substitution of the transformations

$$x = k\,(k+1)\,X\,, z = (k+1)\,P + (2k+1)\,\beta\,, y = (k+1)\,P + (k+1)^2\,\beta \qquad (2.10)$$

in (2.1) leads to the ternary quadratic equation

$$P^2 = X^2 + (2k+1)\beta^2 \tag{2.11}$$

which is satisfied by

$$\beta = 2pq \, , X = (2k+1)p^2 - q^2 \, , P = (2k+1)p^2 + q^2 \tag{2.12}$$

In view of (2.10) , the integer solutions to (2.1) are given by

$$\begin{aligned} x &= k(k+1)[(2k+1)p^2 - q^2] \, , \\ y &= (k+1)[(2k+1)p^2 + q^2] + 2pq\,(k+1)^2 \, , \\ z &= (k+1)[(2k+1)p^2 + q^2] + 2pq\,(2k+1). \end{aligned} \tag{2.13}$$

Note 2

Apart from (2.10), one may consider the transformations as

$$x = k(k+1)X \, , z = (k+1)P - (2k+1)\beta \, , y = (k+1)P - (k+1)^2\beta \, .$$

In this case, (2.1) is satisfied by

$$\begin{aligned} x &= k(k+1)[(2k+1)p^2 - q^2] \, , \\ y &= (k+1)[(2k+1)p^2 + q^2] - 2pq\,(k+1)^2 \, , \\ z &= (k+1)[(2k+1)p^2 + q^2] - 2pq\,(2k+1). \end{aligned}$$

Process 4

Assume

$$z = a^2 + (2k+1)b^2 \tag{2.14}$$

Consider

$$(k+1)^2 = (k + i\sqrt{2k+1})\,(k - i\sqrt{2k+1}) \tag{2.15}$$

Using (2.14) & (2.15) in (2.1) & utilizing technique of factorization gives

$$x + i\sqrt{2k+1}\,y = (k + i\sqrt{2k+1})\,(a + i\sqrt{2k+1}\,b)^2$$

On comparing the coefficients of corresponding terms, consider

$$x = k[a^2 - (2k+1)b^2] - 2(2k+1)ab \,,$$
$$y = 2kab + [a^2 - (2k+1)b^2] \,. \qquad (2.16)$$

Observe that (2.14) & (2.16) satisfy (2.1) .

Process 5

Write (2.1) as

$$x^2 + (2k+1)\, y^2 = (k+1)^2 z^2 * 1 \qquad (2.17)$$

Express the integer 1 on the R.H.S. of (2.17) as

$$1 = \frac{(k + i\sqrt{2k+1})\,(k - i\sqrt{2k+1})}{(k+1)^2} \qquad (2.18)$$

Assume

$$z = (k+1)^2 [a^2 + (2k+1)b^2] \qquad (2.19)$$

Substituting (2.15) ,(2.18) & (2.19) in (2.17) and following the procedure as in Process 4 , we get

$$x = (k+1)\{(k^2 - 2k - 1)\,[a^2 - (2k+1)b^2] - 4k(2k+1)ab\} \,,$$
$$y = (k+1)\{2k[a^2 - (2k+1)b^2] + 2(k^2 - 2k - 1)ab\} \,. \qquad (2.20)$$

Thus , (2.19) & (2.20) satisfy (2.1).

Process 6

It is to be observed that, choosing the values of k to be

$$k = 2s^2 + 2s$$

in (2.1) and employing the transformations

$$x = (2s+1)(2s^2 + 2s + 1)\, X \,,\, y = (2s^2 + 2s + 1)\, Y \,,\, z = (2s+1)\, w \qquad (2.21)$$

in (2.1) , it reduces to the Pythagorean equation given by

$$X^2 + Y^2 = w^2 \qquad (2.22)$$

Considering the most cited solutions of (2.22) and utilizing (2.21) , the solutions in integer for (2.1) are determined.

Deep Science Publishing
https://doi.org/10.70593/978-93-49307-38-4

Chapter 3

A Peer Search on Quaternary Quadratic Diophantine Equation

3.1 Technical Procedure

The homogeneous polynomial equation of degree two with four unknowns to be solved is given by

$$x^2 - xy + y^2 = 4(z^2 - zw + w^2) \quad (3.1)$$

The process of obtaining patterns of integer solutions to (3.1) are illustrated below:

Process 1

Multiplying both sides of (3.1) by 4 and completing the squares, we have

$$\begin{aligned}(2x-y)^2 + 3y^2 &= 4[(2z-w)^2 + 3w^2] \\ &= (4z-2w)^2 + 3(2w)^2\end{aligned} \quad (3.2)$$

Rewrite (3.2) as

$$(2x-y)^2 - (4z-2w)^2 = 3[(2w)^2 - y^2] \quad (3.3)$$

Employing factorization and choosing two non-zero integers α, β ,we write (3.3) as the system of double equations as shown below:

$$\alpha[2x - y + 4z - 2w] = \beta[6w + 3y],$$

$$\beta[2x - y - 4z + 2w] = \alpha[2w - y].$$

Applying the method of cross-multiplication , the integer solutions to (3.1) are given by

$$x = 8\alpha\beta + 12\beta^2 - 4\alpha^2,$$

$$y = -8\alpha^2 - 24\beta^2,$$

$$z = -8\alpha\beta - 12\beta^2 + 4\alpha^2,$$

$$w = 2\alpha^2 + 6\beta^2,$$

Process 2

Taking $x = 2X, y = 2Y$ in (3.1) ,it is written as

$$X^2 - XY + Y^2 = z^2 - zw + w^2$$

Following the analysis of Process 1 , (3.1) is satisfied by

$$x = 8\alpha\beta + 12\beta^2 - 4\alpha^2 ,$$
$$y = -4\alpha^2 - 12\beta^2 ,$$
$$z = -4\alpha\beta - 6\beta^2 + 2\alpha^2 ,$$
$$w = 2\alpha^2 + 6\beta^2 ,$$

Process 3

The option

$$x = p + q, y = p - q, z = r + s, w = r - s, p \neq q \neq r \neq s \neq 0 \quad (3.4)$$

in (3.1) leads to

$$p^2 + 3q^2 = 4(r^2 + 3s^2) \quad (3.5)$$

Consider

$$p + i\sqrt{3}q = (1 + i\sqrt{3})(r + i\sqrt{3}s)$$

Comparing the coefficients of corresponding terms , we have

$$p = r - 3s, q = r + s$$

In view of (3.4) ,it is seen that

$$x = 2(r - s), y = -4s \quad (3.6)$$

Thus , (3.6) & (3.4) satisfy (3.1) .

Process 4

Choosing

$$x = p + q, y = p - q, z = p + s, w = p - s, p \neq q \neq s \neq 0 \quad (3.7)$$

(3.1) represents Pythagorean equation

$$q^2 = p^2 + (2s)^2 \quad (3.8)$$

After some algebra, (3.1) is satisfied by

$$x = 2a^2, y = -2b^2, z = a^2 - b^2 + ab, w = a^2 - b^2 - ab$$

Note 1

It is observed that (3.8) is also satisfied by

$$s = 2(a^2 - b^2), p = 8ab, q = 4(a^2 + b^2)$$

In view of (3.7) , the integer solutions to (3.1) are given by

$$x = 8ab + 4(a^2 + b^2), y = 8ab - 4(a^2 + b^2), z = 8ab + 2(a^2 - b^2),$$
$$w = 8ab - 2(a^2 - b^2)$$

Process 5

Choosing

$$x = p + q\,, y = p - q\,, z = r + q\,, w = r - q\,, p \neq q \neq r \neq 0 \qquad (3.9)$$

in (3.1) ,it reduces to Pythagorean equation

$$p^2 = (3q)^2 + (2r)^2 \qquad (3.10)$$

satisfied by

$$r = 9ab\,, q = 3\,(a^2 - b^2)\,, p = 9(a^2 + b^2)\,, a \neq b \neq 0$$

After simplification, (3.1) is satisfied by

$$x = 12a^2 + 6b^2\,, y = 6a^2 + 12b^2\,, z = 9ab + 3(a^2 - b^2), w = 9ab - 3(a^2 - b^2)$$

Note 2

It is observed that (3.10) is also satisfied by

$$r = 18(a^2 - b^2)\,, q = 24ab\,, p = 36(a^2 + b^2)$$

Using (3.9), (3.1) is satisfied by

$$x = 24ab + 36(a^2 + b^2)\,, y = -24ab + 36(a^2 + b^2), z = 24ab + 18(a^2 - b^2)\,,$$

$$w = -24ab + 18(a^2 - b^2)$$

Process 6

Choosing

$$x = 16c + 160d\,, y = -8c - 16d\,, z = 8R + 4c + 72d\,, w = 8R - 4c - 72d \qquad (3.11)$$

in (3.1) ,it reduces to the ternary quadratic equation

$$c^2 = R^2 + 132d^2 \qquad (3.12)$$

which is satisfied by

$$d = 2ab\,, R = 132a^2 - b^2\,, c = 132a^2 + b^2$$

From (3.11) , the integer solutions to (3.1) are represented by

$$x = 16(132a^2 + b^2) + 320ab\,, y = -8\,(132a^2 + b^2) - 32ab\,,$$

$$z = 144ab + 12*132a^2 - 4b^2\,, w = -144ab + 4*132a^2 - 12b^2$$

Other patterns of integer solutions for (3.1) are as follows:

Express (3.12) as presented in Table 1:

The pair of euations in Table 1:

System	I	II	III	IV	V	VI	VII	VIII	IX	X	XI	XII
c+R=	d^2	$2d^2$	$3d^2$	$6d^2$	$33d^2$	$66d^2$	132d	66d	44d	33d	22d	12d
c-R=	132	66	44	22	4	2	d	2d	3d	4d	6d	11d

After solving the pair of equations for R, c, d and using (3.11), the respective values of x, y, z, w satisfying (3.1) are shown as follows:

Solutions from System I

$$d = 2s, c = 2s^2 + 66, R = 2s^2 - 66$$

$$x = 16(2s^2 + 66) + 320s\ , y = -8(2s^2 + 66) - 32s, z = 8(2s^2 - 66) + 4(2s^2 + 66) + 144s,$$

$$w = 8(2s^2 - 66) - 4(2s^2 + 66) - 144s$$

Solutions from System II

$$c = d^2 + 33, R = d^2 - 33$$

$$x = 16\ (d^2 + 33) + 160d, y = -8(d^2 + 33) - 16d, z = 8(d^2 - 33) + 4(d^2 + 33) + 72d,$$

$$w = 8(d^2 - 33) - 4(d^2 + 33) - 72d$$

Solutions from System III

$$d = 2s, c = 6s^2 + 22, R = 6s^2 - 22$$

$$x = 16(6s^2 + 22) + 320s, y = -8(6s^2 + 22) - 32s, z = 8(6s^2 - 22) + 4(6s^2 + 22) + 144s,$$

$$w = 8(6s^2 - 22) - 4(6s^2 + 22) - 144s$$

Solutions from System IV

$$c = 3d^2 + 11, R = 3d^2 - 11$$

$$x = 16(3d^2 + 11) + 160d, y = -8(3d^2 + 11) - 16d, z = 8(3d^2 - 11) + 4(3d^2 + 11) + 72d,$$

$$w = 8(3d^2 - 11) - 4(3d^2 + 11) - 72d$$

Solutions from System V

$$d = 2s, c = 66s^2 + 2, R = 66s^2 - 2$$

$$x = 16(66s^2 + 2) + 320s, y = -8\ (66s^2 + 2) - 32s, z = 8(66s^2 - 2) + 4(66s^2 + 2) + 144s,$$

$$w = 8(66s^2 - 2) - 4(66s^2 + 2) - 144s$$

Solutions from System VI

$$c = 33d^2 + 1, R = 33d^2 - 1$$

$$x = 16(33d^2 + 1) + 160d, y = -8(33d^2 + 1) - 16d, z = 8(33d^2 - 1) + 4\ 8(33d^2 + 1) + 72d,$$

$$w = 8(33d^2 - 1) - 4\ (33d^2 + 1) - 72d$$

Solutions from System VII

$$d = 2s, c = 133s, R = 131s$$

$$x = 2448s, y = -1096s, z = 1724s, w = 372s$$

Solutions from System VIII

$$c = 34d, R = 32d$$

$$x = 704d, y = -288d, z = 464d, w = 48d$$

Solutions from System IX

$$d = 2s, c = 47s, R = 41s$$

$$x = 1072s, y = -408s, z = 660s, w = -4s$$

Solutions from System X

$$d = 2s, c = 37s, R = 29s$$

$$x = 912s, y = -328s, z = 524s, w = -60s$$

Solutions from System XI

$$c = 14d, R = 8d$$

$$x = 384d, y = -128d, z = 192d, w = -64d$$

Solutions from System XII

$$d = 2s, c = 23s, R = s$$

$$x = 688s, y = -216s, z = 244s, w = -228s$$

Note 3

Apart from (3.11) , there are three more choices of transformations reducing (3.1) to (3.12) which are exhibited below:

Choice 1

$$x = -12c + 104d, y = +8c + 16d, z = 8R - 2c + 60d, w = 8R + 2\ c - 60d$$

Choice 2

$$x = 16c - 160d, y = -8c + 16d, z = 8R + 4c - 72d, w = 8R - 4\ c + 72d$$

Choice 3

$$x = -12c - 104d,\ y = 8c - 16d,\ z = 8R - 2c - 60d,\ w = 8R + 2c + 60d$$

Further, it is seen after some algebra that there are four more choices of transformations reducing (3.1) to $c^2 = X^2 + 132d^2$ which are exhibited below:

Choice 4

$$x = X + 5c + 56d,\ y = -X - 3c - 32d,\ z = X + 4c + 45d,\ w = X + 2c + 21d$$

Choice 5

$$x = X + 5c - 56d,\ y = -X - 3c + 32d,\ z = X + 4c - 45d,\ w = X + 2c - 21d$$

Choice 6

$$x = X - 3c + 32d,\ y = -X + 5c - 56d,\ z = X - 2c + 21d,\ w = X - 4c + 45d$$

Choice 7

$$x = X - 3c - 32d,\ y = -X + 5c + 56d,\ z = X - 2c - 21d,\ w = X - 4c - 45d$$

Following the analysis as presented above in Process 6, many more sets of integer solutions to (3.1) are determined.

Deep Science Publishing
https://doi.org/10.70593/978-93-49307-38-4

Chapter 4

On Non-Homogeneous Ternary Cubic Equation

4.1 Technical Procedure

Consider, the non-homogeneous ternary cubic equation

$$x^3 + x + y^3 + y = 4z^3 \tag{4.1}$$

The option

$$x = u + v, y = u - v, z = u, u \neq v \neq 0 \tag{4.2}$$

in (4.1) leads to the binary quadratic equation

$$u^2 = 3v^2 + 1 \tag{4.3}$$

It is to be noted that (4.3) is the well-known pellian equation whose smallest positive integer solution is

$$v_0 = 1, u_0 = 2$$

If (u_n, v_n) represents the general solution to (4.3), then it is given by

$$u_n + \sqrt{3}\, v_n = (2 + \sqrt{3})^{n+1}, n \geq 0 \tag{4.4}$$

Also,

$$u_n - \sqrt{3}\, v_n = (2 - \sqrt{3})^{n+1} \tag{4.5}$$

Solving (4.4) & (4.5) ,we have

$$u_n = \frac{1}{2} f_n, v_n = \frac{1}{2\sqrt{3}} g_n$$

where

$$f_n = (2+\sqrt{3})^{n+1} + (2-\sqrt{3})^{n+1},$$
$$g_n = (2+\sqrt{3})^{n+1} - (2-\sqrt{3})^{n+1}.$$

From (4.2), (4.1) is satisfied by

$$x_n = \frac{1}{2\sqrt{3}}[\sqrt{3} f_n + g_n],$$
$$y_n = \frac{1}{2\sqrt{3}}[\sqrt{3} f_n - g_n], \qquad (4.6)$$
$$z_n = \frac{1}{2} f_n, n = 0,1,2,...$$

The recurrence relations satisfied by the solutions to (4.1) are given by

$$G_{n+2} - 4G_{n+1} + G_n = 0, n = 0,1,2,...$$

where

$G = x, y, z$ in turn.

The following results are worth to mention:

I. The integer triple (y_n, z_n, x_n) forms an arithmetic progression

II. Replacing n by n+1 in (4.6) , we see that

$$x_{n+1} = \frac{1}{2\sqrt{3}}[3\sqrt{3} f_n + 5g_n],$$
$$y_{n+1} = \frac{1}{2\sqrt{3}}[\sqrt{3} f_n + g_n],$$
$$z_{n+1} = \frac{1}{2}[2 f_n + \sqrt{3} g_n].$$

The following relations are observed:

(a) $x_n = y_{n+1}$

(b) $x_{n+1} y_n - y_{n+1} x_n = 2$

(c) $z_{n+1} y_n - y_{n+1} z_n = 1$

(d) $x_{n+1} z_n - z_{n+1} x_n = 1$

III. Consider $u_n + 2v_n, v_n$ to be the generators of a Pythagorean triangle PT.

Then , it is observed that the difference between its hypotenuse and four times the product of its generators is always equal to 1.

IV. $x_n^3 + y_n^3 + 6x_n y_n z_n = 8z_n^3$

V. $8z_n^2 = 3(x_n^2 + y_n^2) + 2$

VI. $4z_n^2 + 3(x_n y_n - z_n y_n - x_n z_n) = 1$

VII. $3x_n^2 - 6x_n z_n + 2z_n^2 = -1$

Deep Science Publishing
https://doi.org/10.70593/978-93-49307-38-4

Chapter 5

A Peer Search on Ternary Cubic Diophantine Equation

5.1 Technical Procedure

The non-homogeneous Diophantine equation of degree three with three unknowns to be solved is

$$5x^2 - 3y^2 = z^3 \tag{5.1}$$

The process of obtaining various sets of integer solutions to (5.1) through different ways is illustrated below:

Way 1

The choice

$$x = ky, k > 1 \tag{5.2}$$

in (5.1) leads to

$$(5k^2 - 3)y^2 = z^3$$

which is satisfied by

$$y = (5k^2 - 3)t^{3s}, z = (5k^2 - 3)t^{2s} \tag{5.3}$$

In view of (5.2) , we have

$$x = k(5k^2 - 3)t^{3s} \tag{5.4}$$

Thus (5.3) and (5.4) satisfy (5.1).

Way 2

Assumption

$$y = kx, k > 1 \tag{5.5}$$

in (5.1) leads to

$$(5-3k^2)x^2 = z^3$$

which is satisfied by

$$x = (5-3k^2)t^{3s}, z = (5-3k^2)t^{2s} \quad (5.6)$$

In view of (5.5) , we have

$$y = k(5-3k^2)t^{3s} \quad (5.7)$$

Thus , (5.6) and (5.7) represent the integer solutions to (5.1).

Way 3

Taking

$$x = 2X + 6kw, y = 2X + 10kw, z = 2w \quad (5.8)$$

in (1) , we get

$$X^2 = w^2(w + 15k^2) \quad (5.9)$$

which is satisfied by

$$w = (s^2 + 6s - 6)\,k^2, X = (s+3)\,(s^2 + 6s - 6)\,k^3$$

In view of (5.8) , one obtains

$$\begin{aligned} x &= k^3\,(2s+12)\,(s^2+6s-6)\,, \\ y &= k^3\,(2s+16)\,(s^2+6s-6)\,, \\ z &= 2(s^2+6s-6)k^2 \end{aligned} \quad (5.10)$$

which give the integer solutions to (5.1).

Some numerical examples are exhibited in the following Table:

Table- Numerical examples

k	s	X	y	z
1	1	14	18	2
2	3	3024	3696	168
3	4	18360	22032	612
1	2	160	200	20

2	10	39424	44352	1232
2	12	60480	67200	1680

A few interesting relations among the solutions are presented below:

1. $x^3 + 6kxyz + 8k^3z^3 = y^3$

2. The choice

$$s = \alpha^2 + 4\alpha - 3$$

gives

$$y^2 = x^2 + [(2\alpha + 4)kz]^2$$

which is similar to the well-known Pythagorean equation.

3. $\frac{xy}{k^2z^2}$ is one less than a perfect square.

4. Each of the expressions

$$\frac{(y-x)y}{k^2z^2}, \frac{(yz)}{2k}$$

is a perfect square when $s = 2\alpha^2 + 8\alpha$.

5. Each of the expressions

$$\frac{(y-x)x}{k^2z^2}, \frac{(xz)}{2k}$$

is a perfect square when $s = 2\alpha^2 + 4\alpha - 4$.

6. $(y+x)^2 = 4(y-kz)^2 = 4(x+kz)^2$

Way 4

The choice

$$x = X + 48kw, y = X + 80kw, z = 8w \qquad (5.11)$$

in (5.1) , we get

$$X^2 = 256w^2(w + 15k^2) \qquad (5.12)$$

which is satisfied by

$$w=(s^2+6s-6)\ k^2\ , X=16\ (s+3)\ (s^2+6s-6)\ k^3$$

In view of (5.11) , one obtains

$$x=16\ k^3\ (\ s+6)\ (s^2+6s-6)\ ,$$
$$y=16\ k^3\ (\ s+8)\ (s^2+6s-6)\ ,$$
$$z=8\ (s^2+6s-6)\,k^2$$

which give the integer solutions to (5.1).

Way 5

Taking

$$x=2X+6T, y=2X+10T, z=2w \qquad (5.13)$$

in (5.1) , we get

$$X^2-15T^2=w^3 \qquad (5.14)$$

Assume

$$w=a^2-15b^2 \qquad (5.15)$$

Substituting (5.15) in (5.14) and applying factorization ,we consider

$$X+\sqrt{15}\,T=(a+\sqrt{15}\,b)^3$$

from which we get

$$X=f(a,b)\ , T=g(a,b)$$

where

$$f(a,b)=a^3+45ab^2\ ,$$
$$g(a,b)=3a^2b+15b^3$$

In view of (5.13) ,we have

$$x=2f(a,b)+6g(a,b)\ ,$$
$$y=2f(a,b)+10g(a,b)\ ,$$
$$z=2(a^2-15b^2)$$

which satisfy (5.1).

Way 6

Rewrite (5.14) as

$$X^2 - 15T^2 = w^3 * 1 \tag{5.16}$$

Consider the integer 1 on the R.H.S. of (5.16) as

$$1 = (4+\sqrt{15})\ (4-\sqrt{15}) \tag{5.17}$$

Substituting (5.15) & (5.17) in (5.16) and employing the factorization ,one has

$$X + \sqrt{15}\,T = (4+\sqrt{15})\ (a+\sqrt{15}\,b)^3$$

from which we get

$$X = 4f(a,b) + 15g(a,b)\ ,\ T = f(a,b) + 4g(a,b)$$

In view of (5.13) , the corresponding integer solutions to (5.1) are given by

$$x = 14f(a,b) + 54g(a,b)\ ,$$
$$y = 18f(a,b) + 70g(a,b)\ ,$$
$$z = 2(a^2 - 15b^2)\ .$$

Note 1

It is worth to mention that ,in addition to (5.17) ,the following representations for the integer 1 may be considered as shown below:

$$1 = (31+8\sqrt{15})\ (31-8\sqrt{15})\ ,$$
$$1 = \frac{(8+\sqrt{15})\ (8-\sqrt{15})}{49}$$

The above process leads to two more sets of integer solutions to (5.1).

Generation of solutions

Let (x_0, y_0, z_0) be any given integer solution to (5.1).

Assume the second solution to (5.1) as

$$x_1 = h - x_0\ , y_1 = h + y_0\ , z_1 = z_0 \tag{5.18}$$

Substituting (5.18) in (5.1) and simplifying, we have

$$h = 5x_0 + 3y_0$$

Substituting the above value of h in (5.18), we get

$$x_1 = 4x_0 + 3y_0 \, , y_1 = 5x_0 + 4y_0$$

The repetition similarly gives n th solution to (5.1)

$$x_n = \frac{\alpha^n + \beta^n}{2} x_0 + \frac{3(\alpha^n - \beta^n)}{2\sqrt{15}} y_0 \, ,$$

$$v_n = \frac{5(\alpha^n - \beta^n)}{2\sqrt{15}} x_0 + \frac{\alpha^n + \beta^n}{2} y_0 \, ,$$

$$z_n = z_0 \, , n = 1,2,3,...$$

where $\alpha = 4 + \sqrt{15} \, , \beta = 4 - \sqrt{15}$

Deep Science Publishing
https://doi.org/10.70593/978-93-49307-38-4

Chapter 6

On Non-Homogeneous Quaternary Quartic Equation

6.1 Technical Procedure

The quaternary quartic equation under consideration is

$$x^3 + y^3 = 62zw^2 \tag{6.1}$$

The option

$$x = u + v, y = u - v, z = u, u \neq v \neq 0 \tag{6.2}$$

in (6.1) leads to non-homogeneous ternary cubic equation

$$u^2 + 3v^2 = 31w^3 \tag{6.3}$$

The process of solving (6.3) is illustrated below and utilizing (6.2), corresponding solutions to (6.1) are obtained.

Process 1

Assume

$$w = a^2 + 3b^2 \tag{6.4}$$

Express the integer 31 in (6.3) as the product of complex conjugates as below:

$$31 = (2 + i3\sqrt{3})\,(2 - i3\sqrt{3}) \tag{6.5}$$

Substituting (6.4) & (6.5) in (6.3) and employing factorization ,consider

$$u + i\sqrt{3}v = (2 + i3\sqrt{3})\,(a + i\sqrt{3}b)^3 = (2 + i3\sqrt{3})\,[f(a,b) + i\sqrt{3}g(a,b)] \tag{6.6}$$

where

$$f(a,b) = a^3 - 9ab^2 ,$$
$$g(a,b) = 3a^2b - 3b^3 .$$

On comparing the coefficients of corresponding terms in (6.6) ,one obtains

$$u = 2f(a,b) - 9g(a,b)$$
$$v = 3\ f(a,b) + 2\ g(a,b)$$

In view of (6.2) ,we have

$$\begin{aligned} x &= 5f(a,b) - 7g(a,b), \\ y &= -f(a,b) - 11g(a,b), \\ z &= 2f(a,b) - 9g(a,b). \end{aligned} \qquad (6.7)$$

Thus , (6.4) & (6.7) satisfy (6.1).

Note 1

It is to be noted that ,in addition to (6.5), the other representations to 31 are presented below:

$$31 = \frac{(7 + i5\sqrt{3})\ (7 - i5\sqrt{3})}{4},$$
$$31 = \frac{(11 + i\sqrt{3})\ (11 - i\sqrt{3})}{4}.$$

In a similar manner, two choices of solutions for (6.1) are found.

Process 2

Taking

$$v = 3kw \qquad (6.8)$$

in (6.3) , it is written as

$$u^2 = w^2 (31w - 27k^2) \qquad (6.9)$$

After some algebra , it is seen that the R.H.S. of (6.9) is a perfect square for

$$w = 31n^2 + 4kn + k^2 \qquad (6.10)$$

Applying (6.10) in (6.8) ,we have

$$v = 3k(31n^2 + 4kn + k^2) \qquad (6.11)$$

and from (6.9) , it is seen that

$$u = (31n^2 + 4kn + k^2)\ (31n + 2k) \qquad (6.12)$$

Substituting (6.11) & (6.12) in (6.2) ,one obtains

$$\begin{aligned} x &= (31n^2 + 4kn + k^2)\ (31n + 5k)\ , \\ y &= (31n^2 + 4kn + k^2)\ (31n - k)\ , \\ z &= (31n^2 + 4kn + k^2)\ (31n + 2k). \end{aligned} \qquad (6.13)$$

Thus , (6.13) & (6.10) satisfy (6.1) .

Note 2

It is to be noted that, the R.H.S. of (6.9) is also a perfect square when

$$w = 31n^2 - 4kn + k^2$$

For this choice , (6.1) is satisfied by

$$\begin{aligned} x &= (31n^2 - 4kn + k^2)\ (31n + k), \\ y &= (31n^2 - 4kn + k^2)\ (31n - 5k), \\ z &= (31n^2 - 4kn + k^2)\ (31n - 2k). \end{aligned}$$

Process 3

Write (6.3) as

$$u^2 + 3v^2 = 31w^3 * 1 \qquad (6.14)$$

Express the integer 1 in (6.14) as

$$1 = \frac{(1 + i\sqrt{3})\ (1 - i\sqrt{3})}{4} \qquad (6.15)$$

Inserting (6.4) , (6.5) & (6.15) in (6.14) and employing factorization ,consider

$$u + i\sqrt{3}\,v = (2 + i\,3\sqrt{3})\,(a + i\sqrt{3}\,b)^3\,\frac{(1 + i\sqrt{3})}{2}$$

$$= \frac{(-7 + i\,5\sqrt{3})}{2}(a + i\sqrt{3}\,b)^3$$

On equating the coefficients of corresponding terms , we get

$$u = \frac{1}{2}[-7\,f(a,b) - 15g(a,b)]\,,$$

$$v = \frac{1}{2}[5\,f(a,b) - 7\,g(a,b)].$$

In view of (6.2) , we have

$$x = -f(a,b) - 11g(a,b)\,,$$

$$y = -6\,f(a,b) - 4\,g(a,b)\,, \qquad (6.16)$$

$$z = \frac{1}{2}[-7\,f(a,b) - 15\,g(a,b)]\,.$$

Replacing a by $2A$ and b by $2B$ in (6.4) & (6.16), the lattice points for (6.1) are

$$x = 8\,[-f(A,B) - 11g(A,B)],$$

$$y = 8[-6\,f(A,B) - 4\,g(A,B)],$$

$$z = 4[-7\,f(A,B) - 15g(A,B)],$$

$$w = 4[A^2 + 3B^2]\,.$$

Note 3

It is worth to observe that , apart from (6.15) , the integer 1 in (6.14) may be represented as below:

$$1 = \frac{(3r^2 - s^2 + i\sqrt{3}(2rs))\,(3r^2 - s^2 - i\sqrt{3}(2rs))}{(3r^2 + s^2)^2}\,,$$

$$1 = \frac{(r^2 - 3s^2 + i\sqrt{3}(2rs))\,(r^2 - 3s^2 - i\sqrt{3}(2rs))}{(r^2 + 3s^2)^2}\,.$$

Following the above procedure, two more sets of integer solutions to (6.1) are obtained.

Process 4

By inspection, it is seen that (6.3) is satisfied by

$$u = 31^2\, m\,(m^2 + 3n^2)\ ,\qquad v = 31^2\, n\,(m^2 + 3n^2) \tag{6.17}$$

and

$$w = 31(m^2 + 3n^2) \tag{6.18}$$

Using (6.17) in (6.2) ,we have

$$x = 31^2\,(m^2 + 3n^2)\,(m+n)\ ,\qquad y = 31^2\,(m^2 + 3n^2)\,(m-n)\ ,\qquad z = 31^2\, m\,(m^2 + 3n^2)\ . \tag{6.19}$$

Thus , (6.18) & (6.19) satisfy (6.1).

Process 5

Insertion of

$$x = u + w\,,\ y = u - w\,,\ z = 2u \tag{6.20}$$

in (6.1) leads to the binary cubic equation

$$u^2 = w^2\,(62w - 3) \tag{6.21}$$

After some algebra , it is seen that the R.H.S. of (6.21) is a perfect square for

$$w = 62n^2 + 22\,n + 2 \tag{6.22}$$

Applying (6.22) in (6.21) ,we have

$$u = (62n^2 + 22n + 2)\,(62n + 11) \tag{6.23}$$

Substituting (6.22) & (6.23) in (6.20) ,one obtains

$$x = (62n^2 + 22n + 2)\,(62n + 12)\ ,\qquad y = (62n^2 + 22n + 2)\,(62n + 10)\ ,\qquad z = 2\,(62n^2 + 22n + 2)\,(62n + 11). \tag{6.24}$$

Thus , (6.22) & (6.24) satisfy (6.1) .

Note 4

It is to be noted that , the R.H.S. of (6. 21) is also a perfect square when

$$w = 62n^2 - 22n + 2$$

For this choice, (6.1) is satisfied by

$$x = (62n^2 - 22n + 2)\ (62n - 10),$$
$$y = (62n^2 - 22n + 2)\ (62n - 12),$$
$$z = 2(62n^2 - 22n + 2)\ (62n - 11)$$

Process 6

Insertion of

$$x = u + v,\ y = u - v,\ z = 3u \tag{6.25}$$

in (6.1) gives

$$u^2 + 3v^2 = 93\ w^3 \tag{6.26}$$

Assume

$$93 = (9 + i\,2\sqrt{3})\ (9 - i\,2\sqrt{3}) \tag{6.27}$$

Substituting (6.4) & (6.27) in (6.26) and using factorization , consider

$$u + i\sqrt{3}\,v = (9 + i\,2\sqrt{3})\ (f(a,b) + i\sqrt{3}\,g(a,b))$$

from which we have

$$u = 9\ f(a,b) - 6g(a,b)\ ,$$
$$v = 2\ f(a,b) + 9g(a,b)$$

In view of (6.2) , one has

$$\begin{aligned} x &= 11f(a,b) + 3g(a,b), \\ y &= 7f(a,b) - 15g(a,b), \\ z &= 27\ f(a,b) - 18g(a,b). \end{aligned} \tag{6.28}$$

Thus , (6.4) & (6.28) satisfy (6.1).

Process 7

Choosing

$$x = u + 2w, \; y = u - 2w, \; z = 3u \tag{6.29}$$

in (6.1) , it gives

$$u^2 = w^2 (93w - 12) \tag{6.30}$$

After some algebra , it is seen that the R.H.S. of (6.30) is a perfect square for

$$w = 93n^2 + 18n + 1 \tag{6.31}$$

Applying (6.31) in (6.30) ,we have

$$u = (93n^2 + 18n + 1)(93n + 9) \tag{6.32}$$

Substituting (6.31) & (6.32) in (6.29) ,one obtains

$$\begin{aligned} x &= (93n^2 + 18n + 1)(93n + 11), \\ y &= (93n^2 + 18n + 1)(93n + 7), \\ z &= 3(93n^2 + 18n + 1)(93n + 9). \end{aligned} \tag{6.33}$$

Thus , (6.31) & (6.33) satisfy (6.1) .

Note 5

It is to be noted that , the R.H.S. of (6.30) is also a perfect square when

$$w = 93n^2 - 18n + 1$$

For this choice , the solutions to (6.1) are

$$\begin{aligned} x &= (93n^2 - 18n + 1)(93n - 7), \\ y &= (93n^2 - 18n + 1)(93n - 11), \\ z &= 3(93n^2 - 18n + 1)(93n - 9) \end{aligned}$$

Process 8

The option

$$x = u + v, \; y = u - v, \; z = 4u \tag{6.34}$$

in (6.1) gives

$$u^2 + 3v^2 = 124\, w^3 \tag{6.35}$$

Assume

$$124 = (11 + i\sqrt{3})\ (11 - i\ \sqrt{3}) \qquad (6.36)$$

Substituting (6.4) & (6.36) in (6.35) and using factorization, considering

$$u + i\sqrt{3}\,v = (11 + i\ \sqrt{3})\ (f(a,b) + i\sqrt{3}\,g(a,b))$$

we have

$$u = 11\ f(a,b) - 3\ g(a,b)\ ,$$
$$v =\ f(a,b) + 11 g(a,b)$$

In view of (6.2) , one has

$$x = 12\ f(a,b) + 8\ g(a,b)\,,$$
$$y = 10\ f(a,b) - 14\ g(a,b)\,, \qquad (6.37)$$
$$z = 44\ f(a,b) - 12\ g(a,b)\,.$$

Thus , (6.4) & (6.37) satisfy (6.1).

Process 9

Choosing

$$x = u + 2\,w\ , y = u - 2\,w\ , z = 4\,u \qquad (6.38)$$

in (6.1) , it gives

$$u^2 = 4\,w^2\ (31w - 3) \qquad (6.39)$$

After some algebra , it is seen that the R.H.S. of (6.39) is a perfect square for

$$w = 31\ n^2 + 22\ n + 4 \qquad (6.40)$$

Applying (6.40) in (6.39) ,we have

$$u = 2\,(\,31n^2 + 22n + 4)\ (31\ n + 11) \qquad (6.41)$$

Substituting (6.40) & (6.41) in (6.38) ,one obtains

$$x = 2(31n^2 + 22n + 4)(31n + 12),$$
$$y = 2(31n^2 + 22n + 4)(31n + 10), \qquad (6.42)$$
$$z = 8(31n^2 + 22n + 4)(31n + 11).$$

Thus , (6.40) & (6.42) satisfy (6.1) .

Note 6

It is to be noted that , the R.H.S. of (6.39) is also a perfect square when

$$w = 31n^2 - 22n + 4$$

For this choice, (6.1) is satisfied by

$$x = 2(31n^2 - 22n + 4)(31n - 10),$$
$$y = 2(31n^2 - 22n + 4)(31n - 12),$$
$$z = 8(31n^2 - 22n + 4)(31n - 11).$$

Deep Science Publishing
https://doi.org/10.70593/978-93-49307-38-4

Chapter 7

A Peer Search on Quinary Cubic Equation

7.1 Technical Procedure

Consider the homogeneous quinary cubic equation

$$x^3 - y^3 + z^3 + w^3 = k t^3 , k > 0 \tag{7.1}$$

The substitution

$$x = q - p , y = q + p , z = p + r , w = p - r , t = p , q \neq p \neq r \tag{7.2}$$

in (7.1) gives

$$6 r^2 = 6 q^2 + k p^2 \tag{7.3}$$

In what follows, the integer solutions to (7.3) are determined when k takes particular values. In view of (7.2), the corresponding integer solutions to (7.1) are obtained.

Choice 1

The option

$$k = 6 s^2 \tag{7.4}$$

in (7.3) leads to the Pythagorean equation

$$r^2 = q^2 + (sp)^2 \tag{7.5}$$

which is satisfied by

$$r = a^2 + b^2 , q = 2ab , s p = a^2 - b^2 , a > b > 0 \tag{7.6}$$

As integer solutions are required, replacing a by s A and b by s B in (7.6), we have

$$r = s^2 (A^2 + B^2), q = 2 s^2 AB, p = s(A^2 - B^2) \tag{7.7}$$

From (7.2), the solutions for (7.1) are

$$
\begin{aligned}
x &= 2s^2\,AB - s(A^2 - B^2)\ , \\
y &= 2s^2\,AB + s(A^2 - B^2)\ , \\
z &= s(A^2 - B^2) + s^2\,(A^2 + B^2)\ , \\
w &= s(A^2 - B^2) - s^2\,(A^2 + B^2)\ , \\
p &= s(A^2 - B^2)\ .
\end{aligned}
\qquad (7.8)
$$

Note 1

It is worth to mention that (7.5) is also satisfied by

$$r = a^2 + b^2\ , q = a^2 - b^2\ \ , s\,p = 2ab, a > b > 0$$

Replacing a by sA in the above equation, we have

$$r = s^{\,2}A^2 + b^2\ , q = s^{\,2}A^2 - b^2\ \ ,\ p = 2Ab, s\,A > b > 0$$

In this case, the corresponding integer solutions to (7.1) are given by

$$
\begin{aligned}
x &= s^{\,2}A^2 - b^2\ - 2Ab\ , \\
y &= s^{\,2}A^2 - b^2\ + 2Ab\ , \\
z &= 2Ab + (s^{\,2}A^2 + b^2)\ , \\
w &= 2Ab - (s^{\,2}A^2 + b^2)\ , \\
t &= 2Ab.
\end{aligned}
$$

Note 2

Write (7.5) as

$$(sp)^2 + q^2 = r^2 * 1 \qquad (7.9)$$

Assume

$$r = (s\,a)^2 + b^2 \qquad (7.10)$$

Express the integer 1 on the R.H.S. of (7.9) as

$$1 = \frac{(m^2 - n^2 + i\,2mn)\ (m^2 - n^2 - i\,2mn)}{(m^2 + n^2)^2} \qquad (7.11)$$

Substituting (7.10) & (7.11) in (7.9) and utilizing factorization, we consider

$$\begin{aligned} s\,p + i\,q &= (s\,a + i\,b)^2 \,\frac{(m^2 - n^2 + i\,2\,m\,n)}{(m^2 + n^2)} \\ &= (s^2 a^2 - b^2 + i\,2\,s\,a\,b)\,\frac{(m^2 - n^2 + i\,2\,m\,n)}{(m^2 + n^2)} \end{aligned} \qquad (7.12)$$

Equating the real and imaginary parts, we have

$$\begin{aligned} s\,p &= \frac{(s^2 a^2 - b^2)\,(m^2 - n^2) - 4\,s\,a\,b\,m\,n}{(m^2 + n^2)}\,, \\ q &= \frac{2\,s\,a\,b\,(m^2 - n^2) + 2\,m\,n\,(s^2 a^2 - b^2)}{(m^2 + n^2)} \end{aligned} \qquad (7.13)$$

Replacing a by $(m^2 + n^2)\,A$ and b by $(m^2 + n^2)\,B\,s$ in (7.13) & (7.10), we get

$$\begin{aligned} p &= (m^2 + n^2)\,s\{(A^2 - B^2)(m^2 - n^2) - 4\,A\,B\,m\,n\}\,, \\ q &= 2(m^2 + n^2)\,s^2\{\,AB(m^2 - n^2) + (A^2 - B^2)\,m\,n\}\,, \\ r &= (m^2 + n^2)^2\,s^2\,(A^2 + B^2). \end{aligned}$$

From (7.2), one obtains solutions to (7.1).

Choice 2

Let

$$k = 6\,D \qquad (7.14)$$

where D is a non-zero square-free integer. Using (7.14) in (7.3), it becomes

$$r^2 = q^2 + D\,p^2 \qquad (7.15)$$

which is satisfied by

$$p = 2\,a\,b, q = D\,a^2 - b^2, r = D\,a^2 + b^2$$

In view of (7.2), the integer solutions to (7.1) are given by

$$x = Da^2 - b^2 - 2ab,$$
$$y = Da^2 - b^2 + 2ab,$$
$$z = 2ab + Da^2 + b^2,$$
$$w = 2ab - Da^2 - b^2,$$
$$t = 2ab.$$

Note 3

It is to be noted that (7.15) is satisfied by

$$p = 2k, q = (D-1)k, r = (D+1)k \ , D > 1$$

In this case , the integer solutions to (7.1) are given by

$$x = (D-3)k, y = (D+1)k, z = (D+3)k, w = -(D-1)k, t = 2k$$

Note 4

Express (7.15) as the system of double equations given by

$$r + q = Dp^2,$$
$$r - q = 1.$$

Solving the above system of double equations ,we have

$$D = 2m+1, p = 2n+1,$$
$$r = 2(2m+1)(n^2+n) + m + 1,$$
$$q = 2(2m+1)(n^2+n) + m$$

Employing (7.2) ,the corresponding integer solutions to (7.1) are obtained.

Note 5

One may also write (7.15) as the system of double equations as below:

$$r + q = p^2,$$
$$r - q = D.$$

The above system of double equations has two sets of solutions represented by

(i) $D = 2u, p = 2k, r = 2k^2 + u, q = 2k^2 - u$

(ii) $D = 2u+1, p = 2k+1, r = 2k^2 + 2k + u + 1, q = 2k^2 + 2k - u$

Utilizing (7.2), we get two more sets of integer solutions to (7.1).

Choice 3

Taking

$$k = 3 \text{ in (7.3) },$$

we have

$$p^2 = 2(r^2 - q^2) \tag{7.16}$$

which is satisfied by

$$r = 2b^2 + a^2 , q = 2b^2 - a^2 , p = 4ab$$

In view of (7.2), the integer solutions to (7.1) are given by

$$\begin{aligned} x &= 2b^2 - a^2 - 4ab , \\ y &= 2b^2 - a^2 + 4ab , \\ z &= 4ab + 2b^2 + a^2 , \\ w &= 4ab - 2b^2 - a^2 , \\ t &= 4ab. \end{aligned}$$

Choice 4

Rewrite (7.16) as

$$2 r^2 - p^2 = 2q^2 \tag{7.17}$$

Assume

$$q = 2a^2 - b^2 \tag{7.18}$$

The integer 2 on the R.H.S. of (7.17) is expressed as

$$2 = (3\sqrt{2} + 4)(3\sqrt{2} - 4) \tag{7.19}$$

Substituting (7.18) & (7.19) in (7.17) and applying factorization, we consider

$$\sqrt{2}\, r + p = (3\sqrt{2} + 4)(\sqrt{2}\, a + b)^2$$

Equating the coefficients of corresponding terms, we get

$$p = 4(2a^2 + b^2) + 12ab,$$
$$r = 3(2a^2 + b^2) + 8ab \tag{7.20}$$

From (7.2), (7.1) is satisfied by

$$x = 2a^2 - b^2 - 4(2a^2 + b^2) - 12ab\ ,$$
$$y = 2a^2 - b^2 + 4(2a^2 + b^2) + 12ab\ ,$$
$$z = 7(2a^2 + b^2) + 20\ ab\ ,$$
$$w = (2a^2 + b^2) + 4\ ab\ ,$$
$$t = 4(2a^2 + b^2) + 12ab.$$

Choice 5

Write (7.17) as

$$2\ r^2 - p^2 = 2q^2 * 1 \tag{7.21}$$

Consider the integer 1 in (7.21) to be

$$1 = (\sqrt{2} + 1)(\sqrt{2} - 1) \tag{7.22}$$

Substituting (7.18), (7.19) &(7.22) in (7.21) and applying factorization ,consider

$$\sqrt{2}\,r + p = (3\sqrt{2} + 4)\ (\sqrt{2}\,a + b)^2\ (\sqrt{2} + 1)$$
$$= (10 + 7\sqrt{2})\ (2a^2 + b^2 + 2\sqrt{2}\,ab) \tag{7.23}$$

Equating the coefficients of corresponding terms, we get

$$p = 10\ (2a^2 + b^2) + 28ab,$$
$$r = 7(2a^2 + b^2) + 20ab \tag{7.24}$$

From (7.2)

$$\mathrm{x} = (2\mathrm{a}^2 - \mathrm{b}^2) - 10\ (2\mathrm{a}^2 + \mathrm{b}^2) - 28\mathrm{ab}\ ,$$
$$\mathrm{y} = (2\mathrm{a}^2 - \mathrm{b}^2) + 10\ (2\mathrm{a}^2 + \mathrm{b}^2) + 28\mathrm{ab}\ ,$$
$$\mathrm{z} = 17\ (2\mathrm{a}^2 + \mathrm{b}^2) + 48\mathrm{ab}\ ,$$
$$\mathrm{w} = 3\ (2\mathrm{a}^2 + \mathrm{b}^2) + 8\mathrm{ab}\ ,$$
$$\mathrm{t} = 10\ (2\mathrm{a}^2 + \mathrm{b}^2) + 28\mathrm{ab}$$

satisfies (7.1).

Note 6

In addition to (7.22), the integer 1 on the R.H.S. of (7.21) may be expressed as follows:

$$1 = (5\sqrt{2} + 7)\ (5\sqrt{2} - 7)$$

$$1 = \frac{(5\sqrt{2} + 1)\ (5\sqrt{2} - 1)}{49}$$

In a similar manner, two more patterns of solutions to (7.1) are found.

Deep Science Publishing
https://doi.org/10.70593/978-93-49307-38-4

Chapter 8

Techniques to Solve Non-homogeneous Quaternary Quartic Diophantine Equation

8.1 Technical Procedure

The non-homogeneous quaternary quartic diophantine equation to be solved is

$$(x^2 - y^2)^2 + (z^2 - 8)^2 = w^4 \tag{8.1}$$

Assuming

$$x^2 - y^2 = 2ab\,,$$
$$z^2 - 8 = a^2 - b^2 \tag{8.2}$$

in (8.1) , it reduces to the pythagorean equation

$$a^2 + b^2 = w^2 \tag{8.3}$$

Considering the solutions to (8.3) as

$$a = r^2 - 1, b = 2r, w = r^2 + 1 \tag{8.4}$$

we get from (8.2)

$$z = r^2 - 3 \tag{8.5}$$

From (8.2) & (8.4) , we obtain

$$x^2 - y^2 = 4r(r^2 - 1) \tag{8.6}$$

Utilizing the identity

$$(A + 2s)^2 - A^2 = 4s(A + s)$$

in (8.6) ,we get

$$A = \frac{r(r^2 - 1) - s^2}{s} \tag{8.7}$$

For A to be an integer ,choose

$$r = \alpha s \tag{8.8}$$

giving

$$A = \alpha(\alpha^2 s^2 - 1) - s$$

Thus , we obtain the integer solutions to (8.1) as

$$x = \alpha(\alpha^2 s^2 - 1) + s \text{ , } y = \alpha(\alpha^2 s^2 - 1) - s \text{ ,}$$
$$z = (\alpha^2 s^2 - 3) \text{ , } w = (\alpha^2 s^2 + 1).$$

Note 1

Apart from (8) ,we may have

$$r = \alpha s \pm 1 \tag{8.9}$$

From (8.7) , one obtains

$$A = (\alpha s \pm 1)\ \alpha(\alpha s \pm 2) - s$$

Thus , we obtain the integer solutions to (8.1) as

$$x = (\alpha s \pm 1)\ \alpha(\alpha s \pm 2) + s \text{ ,}$$
$$y = (\alpha s \pm 1)\ \alpha(\alpha s \pm 2) - s \text{ ,}$$
$$z = \alpha s\ (\alpha s \pm 2) - 2 \text{ ,}$$
$$w = \alpha s\ (\alpha s \pm 2) + 2$$

In addition to the above two solutions, there are other choices of integer solutions to (8.6) that are illustrated below:

Express (8.6) as Table 1:

Table 1-Simultaneous equations

System	I	II	III	IV	V
x+y	r^3-r	4 r (r+1)	r^2-1	2 (r^3-r)	r (r+1)
x-y	4	r-1	4 r	2	4 (r-1)

Consider System I . Solving the system of double equations, we get

$$x = \frac{(r^3 - r)}{2} + 2, y = \frac{(r^3 - r)}{2} - 2$$

which satisfy (8.1) jointly with z & w given by (8.4) & (8.5) for all values of r.

Consider System II. Solving the system of double equations, we get

$$x = 2r^2 + \frac{(5r - 1)}{2}, y = 2r^2 + \frac{(3r + 1)}{2}$$

As the aim is to obtain integer solutions, taking $r = 2s + 1$, the corresponding integer solutions to (8.1) are given by

$$x = 8s^2 + 13s + 4, y = 8s^2 + 11s + 4,$$
$$z = 4s^2 + 4s - 2, w = 4s^2 + 4s + 2.$$

Consider System III. Solving the pair of equations , we get

$$x = 2r + \frac{(r^2 - 1)}{2}, y = -2r + \frac{(r^2 - 1)}{2}$$

Taking $r = 2s + 1$,the corresponding integer solutions to (8.1) are given by

$$x = 2s^2 + 6s + 2, y = 2s^2 - 2s - 2,$$
$$z = 4s^2 + 4s - 2, w = 4s^2 + 4s + 2$$

Consider System IV. Solving the pair of equations, we get

$$x = r^3 - r + 1, y = r^3 - r - 1$$

which satisfy (8.1) jointly with the values of z & w given by (8.4) & (8.5).

Consider System V. Solving the system of double equations, we get

$$x = \frac{(r^2 + 5r - 4)}{2}, y = \frac{(r^2 - 3r + 4)}{2}$$

which satisfy (8.1) jointly with z & w given by (8.4) & (8.5) for all values of r.

Deep Science Publishing
https://doi.org/10.70593/978-93-49307-38-4

DeepScience
Open Access Books

Chapter 9

On Quaternary Sextic Equation

9.1 Technical Procedure

The quaternary sextic equation under consideration is

$$xy(x+y)=8zw^5 \tag{9.1}$$

Different ways of solving (9.1) are presented below:

Way 1:

Insertion of

$$x=u+v,\ y=u-v,\ z=u \tag{9.2}$$

in (9.1) gives

$$u^2-v^2=4w^5 \tag{9.3}$$

Express (9.3) as in Table 1:

Table 1: Dual equations

Pair	I	II	III	IV	V	VI	VII	VIII
$u+v$	$2w^5$	w^5	w^4	$2w^4$	$4w^4$	$4w^3$	$2w^3$	w^3
$u-v$	2	4	$4w$	$2w$	w	w^2	$2w^2$	$4w^2$

After a few calculations, the solutions satisfying (9.1) are as follows:

Pair I

$$x=2k^5,\ y=2,\ z=k^5+1,\ w=k$$

Pair II

$$x=32k^5,\ y=4,\ z=16k^5+2,\ w=2k$$

Pair III:

$$x = 16k^4, y = 8k, z = 8k^4 + 4k, w = 2k$$

Pair IV:

$$x = 2k^4, y = 2k, z = k^4 + k, w = k$$

Pair V:

$$x = 64k^4, y = 2k, z = 32k^4 + k, w = 2k$$

Pair VI:

$$x = 32k^3, y = 4k^2, z = 16k^3 + 2k^2, w = 2k$$

Pair VII:

$$x = 2k^3, y = 2k^2, z = k^3 + k^2, w = k$$

Pair VIII:

$$x = 8k^3, y = 16k^2, z = 4k^3 + 8k^2, w = 2k$$

Note 1:

Taking

$$v = w^2$$

in (9.3) ,we get after some algebra

$$x = 2k^2(k+1)^3, y = 2k^3(k+1)^2, z = (2k+1)k^2(k+1)^2, w = k(k+1)$$

which satisfy (1).

Way 2 :

The option

$$x = 2u + v, y = 2u - v, z = u \qquad (9.4)$$

in (9.1) gives

$$4u^2 - v^2 = 2w^5 \qquad (9.5)$$

Express (9.5) as in Table 2:

Table 2: Dual equations

System	I	II	III	IV
$2u + v$	w^4	$2w^4$	$2w^3$	$2w^2$

$2u-v$	$2w$	w	w^2	w^3

After a few calculations, the solutions satisfying (9.1) are as follows:

Pair I

$$x = 16k^4, y = 4k, z = 4k^4 + k, w = 2k$$

Pair II :

$$x = 512k^4, y = 4k, z = 128k^4 + k, w = 4k$$

Pair III :

$$x = 16k^3, y = 4k^2, z = 4k^3 + k^2, w = 2k$$

Pair IV :

$$x = 8k^2, y = 8k^3, z = 2k^3 + 2k^2, w = 2k$$

Way 3:

Introduction of the transformations

$$x = 2p, y = 2q, z = k(p+q) \quad (9.6)$$

in (9.1) leads to

$$pq = kw^5 \quad (9.7)$$

Solving (9.7) through different choices and using (9.6), the solutions for (9.1) thus obtained are presented in Table 3 below:

Table 3 : Integer solutions

p	q	w	$x(=2p)$	$y(=2q)$	$z(=k(p+q))$
$k\alpha^5$	1	α	$2k\alpha^5$	2	$k(k\alpha^5+1)$
$k\alpha^4$	α	α	$2k\alpha^4$	2α	$k(k\alpha^4+\alpha)$
$k\alpha^3$	α^2	α	$2k\alpha^3$	$2\alpha^2$	$k(k\alpha^3+\alpha^2)$
$k\alpha^2$	α^3	α	$2k\alpha^2$	$2\alpha^3$	$k(k\alpha^2+\alpha^3)$
$k\alpha$	α^4	α	$2k\alpha$	$2\alpha^4$	$k(k\alpha+\alpha^4)$

$k\alpha^5$	k^5	$k\alpha$	$2k\alpha^5$	$2k^5$	$k(k\alpha^5+k^5)$
$k\alpha^4$	$k^5\alpha$	$k\alpha$	$2k\alpha^4$	$2k^5\alpha$	$k(k\alpha^4+k^5\alpha)$
$k\alpha^3$	$k^5\alpha^2$	$k\alpha$	$2k\alpha^3$	$2k^5\alpha^2$	$k(k\alpha^3+k^5\alpha^2)$
$k\alpha^2$	$k^5\alpha^3$	$k\alpha$	$2k\alpha^2$	$2k^5\alpha^3$	$k(k\alpha^2+k^5\alpha^3)$
$k\alpha$	$k^5\alpha^4$	$k\alpha$	$2k\alpha$	$2k^5\alpha^4$	$k(k\alpha+k^5\alpha^4)$

Some more solution patterns for (9.1) are given below:

Choice 1:

$$x=(8k+1)^5-1, y=1, z=8^4*k^5+5*8^3*k^4+640k^3+80k^2+5k, w=8k+1$$

Choice 2:

$$x=32k^5-4, y=4, z=16k^5-2, w=2k$$

Choice 3:

$$x=16k^4-4k, y=4k, z=4k^4-k, w=2k$$

Choice 4:

$$x=16k^4-8k, y=8k, z=8k^4-4k, w=2k$$

Choice 5:

$$x=8k^3-8k^2, y=8k^2, z=2k^3-2k^2, w=2k$$

Deep Science Publishing
https://doi.org/10.70593/978-93-49307-38-4

Chapter 10

On Non-Homogeneous Ternary Heptic Diophantine Equation

10.1 Technical Procedure

The non-homogeneous heptic Diophantine equation with three unknowns to be solved is given by

$$5(x^2+y^2)-9xy=35z^7 \tag{10.1}$$

Different ways of solving (10.1) are illustrated below:

Way 1:

Taking

$$x=ky \tag{10.2}$$

in (10.1) , it is written as

$$(5k^2-9k+5)\,y^2=35z^7$$

which is satisfied by

$$y=35^4\,(5k^2-9k+5)^3\,s^7\,,\;z=35\,(5k^2-9k+5)\,s^2 \tag{10.3}$$

In view of (10.2) , we get

$$x=35^4\,k\,(5k^2-9k+5)^3\,s^7 \tag{10.4}$$

Thus,(10.3) & (10.4) represent the integer solutions to (10.1).

Way 2:

Introduction of the linear transformations

$$x = (7k-2)v, y = (7k-4)v \tag{10.5}$$

in (10.1) leads to

$$(49k^2 - 42k + 28)v^2 = 35z^7 \tag{10.6}$$

which is satisfied by

$$v = 35^4 (49k^2 - 42k + 28)^3 s^7 , z = 35 (49k^2 - 42k + 28) s^2 \tag{10.7}$$

In view of (10.5) , we have

$$\begin{aligned} x &= 35^4 (7k-2) (49k^2 - 42k + 28)^3 s^7 , \\ y &= 35^4 (7k-4) (49k^2 - 42k + 28)^3 s^7 \end{aligned} \tag{10.8}$$

Thus, (10.7) & (10.8) satisfy (10.1).

Way 3:

Choosing

$$x = u + z^3 , y = u - z^3 \tag{10.9}$$

in (10.1) , it gives

$$u^2 = z^6 (35z - 19) \tag{10.10}$$

After some algebra , it is observed that the R.H.S. of (10.10) is a perfect square when

$$z = z(n) = 35n^2 + 2\alpha_0 n + z_0 \tag{10.11}$$

where (α_0 , z_0) satisfy the equation

$$\alpha^2 = 35z - 19$$

From (10.10) , we get

$$u = (35n + \alpha_0) z^3(n)$$

In view of (10.9) , we have

$$\begin{aligned} x &= x(n) = z^3(n)(35n + \alpha_0 + 1) , \\ y &= y(n) = z^3(n)(35n + \alpha_0 - 1) . \end{aligned} \tag{10.12}$$

It is seen that (10.11) & (10.12) satisfy (10.1) . A few examples are presented in the below Table 10.1:

Table 10.1-Examples

$z(0)$	$x(0)$	$y(0)$
1	5	3
4	$12*4^3$	$10*4^3$
17	$25*17^3$	$23*17^3$
28	$32*28^3$	$30*28^3$

Way 4

Choosing

$$x = 5^4 (u+v), y = 5^4 (u-v), z = 5w \quad (10.13)$$

in (10.1) ,it gives

$$u^2 + 19v^2 = 7w^7 \quad (10.14)$$

Case (i) :

The option

$$v = w^3 \quad (10.15)$$

in (10.14) leads to

$$u^2 = w^6 (7w - 19) \quad (10.16)$$

The R.H.S. of (10.16) is a perfect square when

$$w = w(n) = 7n^2 + 2n\alpha_0 + w_0 \quad (10.17)$$

where (α_0, w_0) satisfy the equation

$$\alpha^2 = 7w - 19$$

From (10.16) ,we get

$$u = (7n + \alpha_0) w^3(n)$$

In view of (10.13) , we have

$$x = x(n) = 5^4 \, w^3(n)(7n + \alpha_0 + 1),$$
$$y = y(n) = 5^4 \, w^3(n)(7n + \alpha_0 - 1), \qquad (10.18)$$
$$z = z(n) = 5\,w(n).$$

It is seen that (10.18) satisfies (10.1). A few examples are presented in the below Table 10.2:

Table 10.2-Examples

$z(0)$	$x(0)$	$y(0)$
5*4	5^4 *4^4	$5^{4*}4^{3*}2$
5^2	5^8	5^7*3
5^2*4	5^4*20^3*12	5^4*20^3*10

Case (ii) :

The option

$$u = 2\ w^3 \qquad (10.19)$$

in (10.14) leads to

$$19v^2 = w^6(7w - 4) \qquad (10.20)$$

The R.H.S. of (10.20) is a perfect square when

$$w = w(n) = 7*19\ n^2 - 8*19n + 44 \qquad (10.21)$$

From (10.20) ,we get

$$v = w^3(7n - 4)$$

In view of (10.13) ,we have

$$x = x(n) = 5^4 \, w^3(n)(7n - 2),$$
$$y = y(n) = 5^4 \, w^3(n)(6 - 7n), \qquad (10.22)$$
$$z = z(n) = 5\,w(n).$$

It is seen that (10.22) satisfies (10.1). A few examples are presented in the below Table 10.3:

Table 10.3-Examples

n	$z(n)$	$x(n)$	$y(n)$
0	5*44	$-2*44^3*5^4$	$6*44^3*5^4$
1	5*25	$5*25^3*5^4$	-25^3*5^4
2	5*272	$12*272^3*5^4$	$-8*272^3*5^4$

Case (iii)

The R.H.S. of (10.20) is a perfect square when

$$w = w(n) = 7*19\, n^2 - 6*19 n + 25 \qquad (10.23)$$

From (10.20) ,we get

$$v = w^3 (7n - 3)$$

In view of (10.13) ,we have

$$\begin{aligned} x &= x(n) = 5^4\, w^3(n)(7n - 1), \\ y &= y(n) = 5^4\, w^3(n)(5 - 7n), \qquad (10.24) \\ z &= z(n) = 5\, w(n). \end{aligned}$$

It is seen that (10.24) satisfies (10.1). A few examples are presented in the below Table 10.4:

Table 10.4-Examples

n	$z(n)$	$x(n)$	$y(n)$
0	5*25	-25^3*5^4	$5*25^3*5^4$
1	5*44	$6*44^3*5^4$	$-2*44^3*5^4$
2	5*329	$13*329^3*5^4$	$-9*329^3*5^4$

Way 5

Choosing

$$x = 7^4 (u + v), y = 7^4 (u - v), z = 7w \qquad (10.25)$$

in (10.1) ,it gives

$$u^2 + 19 v^2 = 5 w^7 \qquad (10.26)$$

Case (iv) :

The option (10.19) in (10.26) leads to

$$19 v^2 = w^6 (5w - 4) \qquad (10.27)$$

The R.H.S. of (10.27) is a perfect square when

$$w = w(n) = 5*19\, n^2 - 6*19 n + 35 \qquad (10.28)$$

From (10.27) ,we get

$$v = w^3 (5n - 3)$$

In view of (10.25) ,we have

$$\begin{aligned} x &= x(n) = 7^4\, w^3(n)(5n - 1), \\ y &= y(n) = 7^4\, w^3(n)(5 - 5n), \\ z &= z(n) = 7\, w(n). \end{aligned} \qquad (10.29)$$

It is seen that (10.29) satisfies (10.1). A few examples are presented in the below Table 10.5:

Table 10.5-Examples

n	z(n)	x(n)	y(n)
0	7*35	-35^3*7^4	$5*35^3*7^4$
1	7*16	$4*16^3*7^4$	0
2	7*187	$9*187^3*7^4$	$-5*187^3*7^4$

Case (v)

The R.H.S. of (10.27) is a perfect square when

$$w = w(n) = 5*19\, n^2 - 4*19 n + 16 \qquad (10.30)$$

From (10.27) ,we get

$$v = w^3 (5n - 2)$$

In view of (10.25) ,we have

$$\begin{aligned} x &= x(n) = 7^4\, w^3(n)(5n), \\ y &= y(n) = 7^4\, w^3(n)(4 - 5n), \\ z &= z(n) = 7\, w(n). \end{aligned} \qquad (10.31)$$

It is seen that (10.31) satisfies (10.1). A few examples are presented in the below Table 10.6:

Table 10.6-Examples

n	$z(n)$	$x(n)$	$y(n)$
0	7*16	0	$4*16^3*7^4$
1	7*35	$5*35^3*7^4$	-35^3*7^4
2	7*244	$10*244^3*7^4$	$-6*244^3*7^4$

Case 6

The option (10.15) in (10.26) leads to

$$u^2 = w^6 (5w - 19) \tag{10.32}$$

The R.H.S. of (10.32) is a perfect square when

$$w = w(n) = 5n^2 - 8n + 7 \tag{10.33}$$

From (10.32) , we have

$$u = w^3 (5n - 4)$$

In view of (10.25) ,we have

$$\begin{aligned} x &= x(n) = 7^4 w^3(n)(5n-3), \\ y &= y(n) = 7^4 w^3(n)(5n-5), \\ z &= z(n) = 7w(n). \end{aligned} \tag{10.34}$$

It is seen that (10.34) satisfies (10.1). A few examples are presented in the below Table 10.7:

Table 10.7-Examples

n	$z(n)$	$x(n)$	$y(n)$
0	7*7	$-3*7^7$	$-5*7^7$
1	7*4	$2*4^3*7^4$	0
2	7*11	$7*11^3*7^4$	$5*11^3*7^4$

Case 7

The R.H.S. of (10.32) is a perfect square when

$$w = w(n) = 5n^2 - 2n + 4 \tag{10.35}$$

From (10.32) , we have

$$u = w^3 (5n - 1)$$

In view of (10.25) ,we have

$$\begin{aligned} x &= x(n) = 7^4 w^3(n)(5n), \\ y &= y(n) = 7^4 w^3(n)(5n-2), \\ z &= z(n) = 7w(n). \end{aligned} \tag{10.36}$$

It is seen that (10.36) satisfies (10.1). A few examples are presented in the below Table 10.8:

Table 10.8-Examples

n	z(n)	x(n)	y(n)
0	7*4	0	$-2*4^3*7^4$
1	7*7	$5*7^3*7^4$	$3*7^7$
2	7*20	$10*20^3*7^4$	$8*20^3*7^4$

Way 6

Choosing

$$x = 35^4(u+v), y = 35^4(u-v), z = 35w \tag{10.37}$$

in (10.1) ,we have

$$u^2 + 19v^2 = w^7 \tag{10.38}$$

Assume

$$w = a^2 + 19b^2 \tag{10.39}$$

Using (10.39) in (10.38) and applying factorization , consider

$$u + i\sqrt{19}\,v = (a + i\sqrt{19}\,b)^7 = f(a,b) + i\sqrt{19}\,g(a,b) \tag{10.40}$$

where

$$f(a,b) = a^7 - 21*19a^5 b^2 + 35*19^2 a^3 b^4 - 7*19^3 b^6 ,$$
$$g(a,b) = 7a^6 b - 35*19a^4 b^3 + 21*19^2 a^2 b^5 - 19^3 b^7 .$$

Equating the real and imaginary parts in (10.40) & from (10.37), (10.1) is satisfied by

$$x = 35^4 [f(a,b) + g(a,b)],$$
$$y = 35^4 [f(a,b) - g(a,b)],$$
$$z = 35(a^2 + 19b^2).$$

Way 7

Write (10.38) as

$$u^2 + 19v^2 = w^7 *1 \qquad (10.41)$$

The integer 1 on the R.H.S. of (10.41) is written as the product of complex conjugates as below:

$$1 = \frac{[(2k^2 - 2k - 9) + i(2k-1)\sqrt{19}][(2k^2 - 2k - 9) - i(2k-1)\sqrt{19}]}{(2k^2 - 2k + 10)^2} \qquad (10.42)$$

Utilizing (10.39) & (10.42) in (10.41) and employing factorization ,consider

$$u + i\sqrt{19}\, v = (a + i\sqrt{19}\, b)^7 \frac{[(2k^2 - 2k - 9) + i(2k-1)\sqrt{19}]}{(2k^2 - 2k + 10)}$$
$$= [f(a,b) + i\sqrt{19}\, g(a,b)] \frac{[(2k^2 - 2k - 9) + i(2k-1)\sqrt{19}]}{(2k^2 - 2k + 10)}$$

Equating the real and imaginary parts in the above equation ,we have

$$u = \frac{1}{(2k^2 - 2k + 10)} [(2k^2 - 2k - 9) f(a,b) - 19(2k-1) g(a,b)] ,$$
$$v = \frac{1}{(2k^2 - 2k + 10)} [(2k-1) f(a,b) + (2k^2 - 2k - 9)\, g(a,b)] . \qquad (10.43)$$

As the aim is to obtain integer solutions, replacing a by $(2k^2 - 2k + 10)A$ and

b by $(2k^2 - 2k + 10)B$ in (10.43) ,(10.39) and using (10.37) , the corresponding integer solutions to (10.1) are obtained.

It is worth to mention here that, in addition to (10.42) ,one may consider the integer 1 as below

$$1=\frac{[19r^2-s^2+i\sqrt{19}(2rs)]\,19r^2-s^2-i\sqrt{19}(2rs)]}{(19r^2+s^2)^2}\ ,$$

$$1=\frac{[\ r^2-19s^2+i\sqrt{19}(2rs)]\ \ r^2-19s^2-i\sqrt{19}(2rs)]}{(\ r^2+19s^2)^2}$$

Following the procedure as above, one obtains solutions to (10.1).

Deep Science Publishing
https://doi.org/10.70593/978-93-49307-38-4

Chapter 11

On Non-Homogeneous Quinary Heptic Equation

11.1 Technical Procedure

Consider the quinary heptic equation

$$x^4 - y^4 = 41(z^2 - w^2)p^5 \tag{11.1}$$

Introducing the linear transformations

$$x = u + v,\ y = u - v,\ z = 2u + v \text{ and } w = 2u - v \tag{11.2}$$

in (11.1), it leads to

$$u^2 + v^2 = 41p^5 \tag{11.3}$$

PATTERN-1

Assume $p = a^2 + b^2$ (11.4)

where a and b are non-zero distinct integers.

Write 41 as $41 = (4+5i)(4-5i)$ (11.5)

Using (11.4) & (11.5) in (11.3) and employing the method of factorization, define

$$u + iv = (4+5i)(a+ib)^5 \tag{11.6}$$

Equating the real and imaginary parts of (11.6), we get

$$\begin{aligned} u &= 4a^5 - 25a^4b - 40a^3b^2 + 50a^2b^3 + 20ab^4 - 5b^5 \\ v &= 5a^5 + 20a^4b - 50a^3b^2 - 40a^2b^3 + 25ab^4 + 4b^5 \end{aligned} \tag{11.7}$$

Substituting (11.7) in (11.2), we have

$$\begin{aligned} x(a,b) &= 9a^5 - 5a^4b - 90a^3b^2 + 10a^2b^3 + 45ab^4 - b^5 \\ y(a,b) &= -a^5 - 45a^4b + 10a^3b^2 + 90a^2b^3 - 5ab^4 - 9b^5 \\ z(a,b) &= 13a^5 - 30a^4b - 130a^3b^2 + 60a^2b^3 + 65ab^4 - 6b^5 \\ w(a,b) &= 3a^5 - 70a^4b - 30a^3b^2 + 140a^2b^3 + 15ab^4 - 14b^5 \\ p(a,b) &= a^2 + b^2 \end{aligned} \qquad (11.8)$$

satisfying (11.1) .

PROPERTIES:

- $a[x(a,a) + y(a,a) + z(a,a) + w(a,a)]$ is a Nasty number.
- $x(a,1) + y(a,1,) - 100t_{3,a^2} + 150t_{4,a} \equiv -10(\mathrm{mod}8)$
- $-x(a,a) + y(a,a) + z(a,a) - w(a,a) + 2p(a,a)$ is a perfect square.

Remark 1:

Choosing

$$z = 2uv + 1, w = 2uv - 1 \qquad (11.9)$$

In (11.2), one obtains

$$\begin{aligned} z(a,b) &= 2(4a^5 - 25a^4b - 40a^3b^2 + 50a^2b^3 + 20ab^4 - 5b^5) \\ &\quad *(5a^5 + 20a^4b - 50a^3b^2 + 40a^2b^3 + 25ab^4 + 4b^5) + 1 \\ w(a,b) &= 2(4a^5 - 25a^4b - 40a^3b^2 + 50a^2b^3 + 20ab^4 - 5b^5) \\ &\quad *(5a^5 + 20a^4b - 50a^3b^2 + 40a^2b^3 + 25ab^4 + 4b^5) - 1 \end{aligned} \qquad (11.10)$$

satisfying (11.1) jointly with (11.8).

Remark 2:

Taking $z = uv + 2, w = uv - 2$ (11.11)

In (11.2), one obtains

$$\begin{aligned} z(a,b) &= (4a^5 - 25a^4b - 40a^3b^2 + 50a^2b^3 + 20ab^4 - 5b^5) \\ &\quad *(5a^5 + 20a^4b - 50a^3b^2 + 40a^2b^3 + 25ab^4 + 4b^5) + 2 \\ w(a,b) &= (4a^5 - 25a^4b - 40a^3b^2 + 50a^2b^3 + 20ab^4 - 5b^5) \\ &\quad *(5a^5 + 20a^4b - 50a^3b^2 + 40a^2b^3 + 25ab^4 + 4b^5) - 2 \end{aligned} \qquad (11.12)$$

satisfying (11.1) jointly with (11.8).

PATTERN-2:

Instead of (11.5),

41 can also be written as $41 = (5+4i)(5-4i)$

By using (11.4) and (11.13) in (11.3) and applying the same procedure in pattern - 1, the corresponding integer solutions to (11.1) are found to be

$$x(a,b) = 9a^5 + 5a^4b - 90a^3b^2 - 10a^2b^3 + 45ab^4 + b^5$$
$$y(a,b) = a^5 - 45a^4b - 10a^3b^2 + 90a^2b^3 + 5ab^4 - 9b^5$$
$$z(a,b) = 14a^5 - 15a^4b - 140a^3b^2 + 30a^2b^3 + 70ab^4 - 3b^5$$
$$w(a,b) = 6a^5 - 65a^4b - 60a^3b^2 + 130a^2b^3 + 30ab^4 - 13b^5$$
$$p(a,b) = a^2 + b^2$$

PROPERTIES:

- $324[x(a,a) + y(a,a) + z(a,a) + w(a,a)]$ is a quintic integer.
- $x(a,b) + y(a,b) - 10(pr_a)^5 \equiv 2 \pmod{10}$
- $-x(a,a) + y(a,a) + z(a,a) - w(a,a) = 0$.

Remark 3:

Considering (11.9) & (11.11), we have

SET I:

$$\begin{aligned} z(a,b) &= 2(5a^5 - 20a^4b - 50a^3b^2 + 40a^2b^3 + 25ab^4 - 4b^5) \\ &\quad *(4a^5 + 25a^4b - 40a^3b^2 - 50a^2b^3 + 20ab^4 + 5b^5) + 1 \\ w(a,b) &= 2(5a^5 - 20a^4b - 50a^3b^2 + 40a^2b^3 + 25ab^4 - 4b^5) \\ &\quad *(4a^5 + 25a^4b - 40a^3b^2 - 50a^2b^3 + 20ab^4 + 5b^5) - 1 \end{aligned}$$

SET II:

$$z(a,b) = (5a^5 - 20a^4b - 50a^3b^2 + 40a^2b^3 + 25ab^4 - 4b^5)$$
$$*(4a^5 + 25a^4b - 40a^3b^2 - 50a^2b^3 + 20ab^4 + 5b^5) + 2$$
$$w(a,b) = (5a^5 - 20a^4b - 50a^3b^2 + 40a^2b^3 + 25ab^4 - 4b^5)$$
$$*(4a^5 + 25a^4b - 40a^3b^2 - 50a^2b^3 + 20ab^4 + 5b^5) - 2$$

Considering (11.2) together with sets I and II in turn, one obtains two more patterns of solutions to (11.1).

PATTERN-3:

One may write (11.3) as $u^2 + v^2 = 41p^5 * 1$ (11.13)

Write 1 as $1 = \frac{(1 = 4 + 3i)(4 - 3i)}{25}$ (11.14)

Using (11.4), (11.5) & (11.14) in (11.13) and applying the method of factorization, define

$$u + iv = \frac{(4+3i)(4+5i)}{5}(a+ib)^5$$

Equating the real and imaginary parts, we get

$$u = \frac{(a^5 - 160a^4b - 10a^3b^2 + 320a^2b^3 + 5ab^4 - 32b^5)}{5}$$
$$v = \frac{(32a^5 + 5a^4b - 320a^3b^2 - 10a^2b^3 + 160ab^4 + b^5)}{5} \quad (11.15)$$

As our interest is on finding integer solutions, choose $a = 5A, b = 5B$ in (11.15), one obtains

$$x(A,B) = 20625A^5 - 96875A^4B - 206250A^3B^2 + 193750A^2B^3 + 103125AB^4 - 19375B^5$$
$$y(A,B) = -1937A^5 - 103125A^4B + 193750A^3B^2 + 206250A^2B^3 - 96875AB^4 + 20025B^5$$
$$z(A,B) = 21250A^5 - 203125A^4B - 215200A^3B^2 + 393750A^2B^3 + 106250AB^4 - 39375B^5$$
$$w(A,B) = -18750A^5 - 203125A^4B - 187500A^3B^2 + 406250A^2B^3 - 93750AB^4 - 40625B^5$$
$$p(A,B) = 25A^2 + 25B^2$$
(11.16)

satisfying (11.1).

Remark 4:

Considering (11.9) & (11.11), we have

SET I:

$$z(A,B)=2(625A^5-100000A^4B-6250A^3B^2+200000A^2B^3+3125AB^4-20000b^5)$$
$$(20000A^5+3125A^4B-200000A^3B^2-6250A^2B^3+100000AB^4+625B^5)+1$$
$$w(A,B)=2(625A^5-100000A^4B-6250A^3B^2+200000A^2B^3+3125AB^4-20000b^5)$$
$$(20000A^5+3125A^4B-200000A^3B^2-6250A^2B^3+100000AB^4+625B^5)-1$$

SET II:

$$z(A,B)=(625A^5-100000A^4B-6250A^3B^2+200000A^2B^3+3125AB^4-20000b^5)$$
$$(20000A^5+3125A^4B-200000A^3B^2-6250A^2B^3+100000AB^4+625B^5)+2$$
$$w(A,B)=(625A^5-100000A^4B-6250A^3B^2+200000A^2B^3+3125AB^4-20000b^5)$$
$$(20000A^5+3125A^4B-200000A^3B^2-6250A^2B^3+100000AB^4+625B^5)-2$$

Considering (11.16) together with sets I and II in turn, one obtains two more patterns of solutions to (11.1).

PATTERN-4:

Introduction of the transformations

$$x=k\,y\;,z=k\,w\;,k>1 \tag{11.18}$$

in (11.1) leads to

$$(k^2+1)\,y^4=41w^2P^5 \tag{11.19}$$

which is satisfied by

$$\begin{aligned} y&=41^2(k^2+1)^s\,\alpha^{3\beta}\,,\\ w&=41(k^2+1)^{2s-2}\,\alpha^{\beta}\,,\\ P&=41(k^2+1)\,\alpha^{2\beta}\;\;,\alpha>1,s\geq 1 \end{aligned} \tag{11.20}$$

In view of (11.18) , we have

$$x = 41^2 k(k^2+1)^s \alpha^{3\beta},$$
$$z = 41k(k^2+1)^{2s-2} \alpha^{\beta} \qquad (11.21)$$

Thus , (11.20) and (11.21) represent the integer solutions to (11.1).

3. REMARKABLE OBSERVATIONS:

Employing the integral solutions of (11.1) and (11.3) the following expressions among the special polygonal & pyramidal numbers are given below

1. $\left[\frac{3P_x^3}{t_{3,x+1}}\right]^2 + \left[\frac{12p_y^5}{s_{y+1}-1}\right]^2 \equiv 0(\bmod 41)$.

2. $\left[\frac{4Pt_{x-3}}{P_{x-3}^3}\right]^4 - 41\left[\left[\frac{3P_z^3}{t_{3,z+1}}\right]^2 - \left[\frac{P_w^5}{t3,w}\right]^2\right]\left[\frac{4P_p^5}{ct_{4,p}-1}\right]^5$ is a bi quadratic integer.

3. $41\left[\left[\frac{6P_{z-2}^3}{\Pr_{z-2}}\right]^2 - \left[\frac{4Pt_{w-3}}{P_{w-3}^3}\right]^2\right][4\Pr_p+1]^2 + \left[\frac{P_y^5}{t_{3,y}}\right]^4$ is a perfect square.

4. $\left[\frac{4P_u^5}{Ct_{4,u}-1}\right]^2 - \left[\frac{12p_v^5}{s_{v+1}-1}\right]^2 \equiv 0(\bmod 41)$.

5. $41^4\left\{\left[\frac{3P_{u-2}^3}{t_{3,u-2}}\right]^2 - \left[\frac{P_v^5}{\Pr_v}\right]^2\right\}$ is a quintic integer.

https://doi.org/10.70593/978-93-49307-38-4

Chapter 12

On The Ternary Surd Equation

12.1 Technical Procedure

Consider the ternary surd equation

$$x+\sqrt{x}+y+\sqrt{y}=z+\sqrt{z} \tag{12.1}$$

The substitution

$$x=u^2, y=v^2, z=w^2 \tag{12.2}$$

in (12.1) leads to the ternary quadratic equation

$$u^2+u+v^2+v=w^2+w \tag{12.3}$$

Two different methods of solving (12.3) are illustrated below:

Method 1

Write (12.3) as

$$(2u+1)^2+(2v+1)^2=(2w+1)^2+1=[2w+1)^2+1]*1 \tag{12.4}$$

Assume

$$1=\frac{(3+4i)(3-4i)}{25} \tag{12.5}$$

Substituting (12.5) in (12.4) and applying factorization, consider

$$(2u+1)+i(2v+1)=[(2w+1)+i]*\frac{(3+4i)}{5}$$

Equating the coefficients of corresponding terms and taking

$$w=5s+1,$$

one obtains

$$x = x(s) = 9s^2, y = y(s) = (4s+1)^2, z = z(s) = (5s+1)^2 \qquad (12.6)$$

satisfying (12.1).

A few interesting relations satisfied by the solutions (12.6) of (12.1) are shown below:

- $[z(s) - x(s) - y(s) + 1]^2 = 8t_{3,s} + 1$.
- $s = t^2 \pm t \Rightarrow y(s)$ is an integer raised to 4th power.
- $s = 5k^2 \mp 2k \Rightarrow z(s)$ is a biquadratic integer.
- $[y(s) - 1]^2 = 16(z - x - y)^2 (8t_{3,s} + 1)$.
- $9(z - x - y)^2 = (4s^2)9 = 4x$
- $12x(z - x - y) = 216s^3$, a cubical integer.
- $5x + 5y - 4z - 1$ is a perfect square.
- $9x + 10y - 8z - 2 = 41s^2 = s^2 + 4s^2 + 36s^2$ is a sum of three squares.
- $9x + 10y - 8z - 2 = 41s^2 = (4s)^2 + (5s)^2$ is a sum of two squares.

From the given solutions, one may generate second order Ramanujan numbers.

Illustration:

Let

$$x = x(s) = 9s^2 * 1 = 9s * s = s^2 * 9$$

I II III

- $\text{I, II} \Rightarrow (9s^2 + 1)^2 + (9s - s)^2 = (9s^2 - 1)^2 + (9s + s)^2 = 81s^4 + 82^2 + 1$

- I, III $\Rightarrow (9s^2+1)^2+(s^2-9)^2=(9s^2-1)^2+(s^2+9)^2=82s^4+82$
- II, III $\Rightarrow (9s+s)^2+(s^2-9)^2=(9s-s)^2+(s^2+9)^2=s^4+82s^2+81$

Thus, $81s^4+82^2+1$, $82s^4+82$, s^4+82s^2+81 represent second order Ramanujan Numbers.

Method 2

Taking

$$U=2u+1\,,V=2v+1\,,W=2w+1 \tag{12.7}$$

in (4) , we have

$$U^2+V^2=W^2+1 \tag{12.8}$$

Choosing

$$W=k\ V\ ,k>1 \tag{12.9}$$

in (12.8) ,we get

$$U^2=(k^2-1)\ V^2+1$$

which is the well-known Pell equation whose general solution $(U_n\,,V_n)$ is given by

$$U_n=\frac{f_n}{2}\,,V_n=\frac{g_n}{2\sqrt{k^2-1}} \tag{12.10}$$

where

$$f_n=(k+\sqrt{k^2-1})^{n+1}+(k-\sqrt{k^2-1})^{n+1}\,,$$
$$g_n=(k+\sqrt{k^2-1})^{n+1}-(k-\sqrt{k^2-1})^{n+1}$$

In view of (12.7), it is seen that

$$u_n = \frac{(f_n - 2)}{4}\ , v_n = \frac{(g_n - 2\sqrt{k^2 - 1})}{4\sqrt{k^2 - 1}}\ , w_n = \frac{(k\ g_n - 2\sqrt{k^2 - 1})}{4\sqrt{k^2 - 1}}$$

After some algebra, it is observed that the above values of u_n, v_n, w_n are integers when n is even & k is odd and substituting in (12.2), the solutions for (12.1) are obtained.

Deep Science Publishing
https://doi.org/10.70593/978-93-49307-38-4

Chapter 13

On The Quinary Surd Equation

13.1 Technical Procedure

The quinary surd equation under consideration is

$$u\sqrt[3]{x^2+y^2}+v\sqrt[3]{X^2+Y^2}=\left(k^2+s^2\right)R^2 \quad ;k,s\neq 0 \tag{13.1}$$

The assumption

$$\rho^3=x^2+y^2 \tag{13.2}$$

is satisfied by

$$x=m\left(m^2+n^2\right),\ y=n\left(m^2+n^2\right). \tag{13.3}$$

Similarly, the choices

$$\mathrm{X}=\mathrm{m}\left(\mathrm{m}^2-3\mathrm{n}^2\right),\ Y=n\left(3m^2-n^2\right) \tag{13.4}$$

satisfy

$$\tau^3=\mathrm{X}^2+\mathrm{Y}^2.$$

On substituting (13.3), (13.4) in (13.1), we obtain

$$(u+v)(m^2+n^2)=\left(k^2+s^2\right)R^2. \tag{13.5}$$

Various choices of solutions in integers to (13.5) are illustrated as follows. Using (13.3) & (13.4), one obtains solutions for (13.1).

Set 1.

Choose *u* & *v* such that

$$u+v=\alpha^2 \tag{13.6}$$

write

$$\mathrm{R} = \alpha \mathrm{S}. \tag{13.7}$$

The substitution of (13.6), (13.7) in (13.5) leads to,

$$(m^2 + n^2) = \left(k^2 + s^2\right) S^2 \tag{13.8}$$

Let

$$S = A^2 + B^2. \tag{13.9}$$

Using (13.9) in (13.8) and applying the method of factorization, consider $m + in = (k + is)(A + iB)^2$.

From which, observe

$$m = k\left(A^2 - B^2\right) - 2sAB \ , \ n = k\left(A^2 - B^2\right) - 2kAB. \tag{13.10}$$

Using (13.9) and (13.10) in (13.8), (13.3) & (13.4), the positive solutions in integer solutions of (13.1) are represented by

$x = [\,k\left(A^2 - B^2\right) - 2sAB\,][\,(k^2 + s^2)(A^2 + B^2)^2\,]$,

$y = [\,s\left(A^2 - B^2\right) + 2kAB\,][\,(k^2 + s^2)(A^2 + B^2)^2\,]$,

$X = [\,k\left(A^2 - B^2\right) - 2sAB\,][\,(k^2 - 3s^2)(A^2 - B^2)^2 + 4A^2B^2\left(s^2 - 3k^2\right) - 16ksAB\left(A^2 - B^2\right)]$
,

$Y = [\,s\left(A^2 - B^2\right) + 2kAB\,][\,(3k^2 - s^2)(A^2 - B^2)^2 + 4A^2B^2\left(3s^2 - k^2\right) - 16ksAB\left(A^2 - B^2\right)]$

, $R = \alpha\left(A^2 + B^2\right)$.

Properties.

1) $x\left(k, -2\alpha^2, A, A\right)$ *is expressed as sum of two squares.*

2) $X\left(k, 3u^2, A, A\right) - x\left(k, 3u^2, A, A\right)$ *represents the area of Pythagorean Triangle.*

3) $X\left(k, -3u^2, A, A\right) + 3x\left(k, -3u^2, A, A\right)$ *represents a Nasty number.*

4) $X(k, k+1, A, A) - x(k, k+1, A, A)$ *represent a square multiple of Pentagonal Pyramidal number of rank k.*

5) $X\left(1, 3n^2 - n, A, A\right) - x\left(1, 3n^2 - n, A, A\right) + 16A^6$ *represents a square multiple of star number of rank n.*

6) $2[y(2n-1, s, A, A) + Y(2n-1, s, A, A)]$ *represents a square multiple of Gnomonic number of rank n.*

7) $y\left(3k^2, s, A, A\right) + Y\left(3k^2, s, A, A\right)$ *represents a Nasty number.*

Set 2.

Choose u and v such that

$$u + v = \alpha^2 + \beta^2 \tag{13.11}$$

Substituting (13.11) in (13.5), we get

$$\left(\alpha^2 + \beta^2\right)\left(m^2 + n^2\right) = \left(k^2 + s^2\right)R^2 . \tag{13.12}$$

Take

$$R = A^2 + B^2 . \tag{13.13}$$

Using (13.13) in (13.12) and employing factorization, one has

$$(\alpha + i\beta)(m + in) = (k + is)(A + iB)^2 = (k + is)\left(A^2 - B^2 + i2AB\right).$$

Comparing the corresponding terms, one has

$$m\alpha - n\beta = k\left(A^2 - B^2\right) - 2sAB.$$
$$n\alpha + m\beta = s\left(A^2 - B^2\right) + 2kAB.$$

Solving the above two equations,

$$m\left(\alpha^2 + \beta^2\right) = \beta[\,s\left(A^2 - B^2\right) + 2kAB\,] + \alpha[\,k\left(A^2 - B^2\right) - 2sAB\,],$$
$$n\left(\alpha^2 + \beta^2\right) = \alpha[\,s\left(A^2 - B^2\right) + 2kAB\,] - \beta[\,k\left(A^2 - B^2\right) - 2sAB. \tag{13.14}$$

Choosing $A = \left(\alpha^2 + \beta^2\right)P$ and $B = \left(\alpha^2 + \beta^2\right)Q$ in (13.13) and (13.14), one has

$$R = \left(\alpha^2 + \beta^2\right)^2\left(P^2 + Q^2\right) \tag{13.15}$$

$$m = \left(\alpha^2 + \beta^2\right)\left(\beta f(P,Q,k,s) + \alpha g(P,Q,k,s)\right)$$
$$n = \left(\alpha^2 + \beta^2\right)\left(\alpha f(P,Q,k,s) - \beta g(P,Q,k,s)\right)$$

Substituting the values of m and n in (13.3) and (13.4), we have

$$x = (\alpha^2 + \beta^2)^4 (\beta f(P,Q,k,s) + \alpha g(P,Q,k,s))[(f^2(P,Q,k,s) + g^2(P,Q,k,s))]$$
$$y = (\alpha^2 + \beta^2)^4 (\alpha f(P,Q,k,s) - \beta g(P,Q,k,s))[(f^2(P,Q,k,s) + g^2(P,Q,k,s))]$$
$$X = (\alpha^2 + \beta^2)^3 (\beta f(P,Q,k,s) + \alpha g(P,Q,k,s))[\begin{pmatrix} f^2(P,Q,k,s)(\beta^2 - 3\alpha^2) \\ + g^2(P,Q,k,s)(\alpha^2 - 3\beta^2) \end{pmatrix}] \quad (13.16)$$
$$Y = (\alpha^2 + \beta^2)^3 (\alpha f(P,Q,k,s) - \beta g(P,Q,k,s))[\begin{pmatrix} f^2(P,Q,k,s)(3\beta^2 - \alpha^2) \\ + g^2(P,Q,k,s)(3\alpha^2 - \beta^2) \end{pmatrix}],$$

where

$$f(P,Q,k,s) = s(P^2 - Q^2) + 2kPQ \; ; \; g(P,Q,k,s) = k(P^2 - Q^2) - 2sPQ \, .$$

Note that (13.1) is satisfied by (13.15) and (13.16).

Set 3.

(13.8) can be written as

$$(m^2 + n^2) = (k^2 + s^2) S^2 * 1 \quad (13.17)$$

Assume 1 as

$$1 = \frac{(p^2 - q^2 + i2pq)(p^2 - q^2 - i2pq)}{(p^2 + q^2)^2} \quad (13.18)$$

Using (13.18) and (13.9) in (13.17) and applying method of factorization, consider

$$m + in = (k + is) \frac{(p^2 - q^2 + i2pq)}{(p^2 + q^2)} (A^2 - B^2 + i2AB).$$

$$m + in = \frac{(k + is)}{(p^2 + q^2)} [\{(p^2 - q^2)(A^2 - B^2) - 4pqAB\} + i\{2pq(A^2 - B^2) + 2AB(p^2 - q^2)\}].$$

Following the procedure as in set 2, (13.1) is satisfied by

$$x = (p^2 + q^2)^3 (kf - sg)(f^2 + g^2)(k^2 + s^2),$$
$$y = (p^2 + q^2)^3 (sf + kg)(f^2 + g^2)(k^2 + s^2),$$
$$X = (p^2 + q^2)^3 (kf - sg)[f^2(k^2 - 3s^2) + g^2(s^2 - 3k^2)],$$
$$Y = (p^2 + q^2)^3 (sf + kg)[f^2(3k^2 - s^2) + g^2(3s^2 - k^2)],$$
$$R = \alpha(p^2 + q^2)^2 (a^2 + b^2),$$

where

$$\mathrm{f}(\mathrm{a},\mathrm{b},\mathrm{p},\mathrm{q}) = (\mathrm{p}^2 - \mathrm{q}^2)(\mathrm{a}^2 - \mathrm{b}^2) - 4\mathrm{abpq}\ ;$$
$$\mathrm{g}(\mathrm{a},\mathrm{b},\mathrm{p},\mathrm{q}) = 2\mathrm{pq}(\mathrm{a}^2 - \mathrm{b}^2) + 2\mathrm{ab}(\mathrm{p}^2 - \mathrm{q}^2).$$

Conclusion:

The authors have taken care to present the process of obtaining real integer solutions in an elegant way for special multivariable quadratic Diophantine equations. The researchers in this field may improve their ability to specialize and generalize, to pose and solve meaningful problems, to look for patterns and relations and to apply the logical thinking behind mathematical process. The readers in this field may be motivated in determining other patterns of solutions to the considered problems.

References:

Carmichael, R.D. (1959). The theory of numbers and Diophantine Analysis, *Dover publications,* New York.
https://www.scirp.org/reference/referencespapers?referenceid=3426822

Dickson, L.E.(1952). History of Theory of Numbers, Vol.11, *Chelsea Publishing company*, New York.
https://archive.org/details/historyoftheoryo01dick/page/n1/mode/2up

Gopalan M.A. et.al. (2012a). Integral Points on the Non- homogeneous Cone $2z^2 + 4xy + 8x - 4z + 2 = 0$, *Global Journal of Mathematics and Mathematical Sciences*, Vol. 2(1), Pp. 61-67.
https://www.ripublication.com/Volume/gjmmsv2n1.htm

Gopalan M.A. et.al. (2015b).Observations on the cone $z^2 = ax^2 + a(a-1)y^2$, *International Journal of Multidisciplinary Research and Development,* Vol.2 (9), Pp.304-305.
https://www.allsubjectjournal.com/assets/archives/2015/vol2issue9/164.pdf

Gopalan M.A. et.al. (2024c) A class of new solutions in integers to Ternary Quadratic Diophantine Equation $12(x^2 + y^2) - 23xy + 2x + 2y + 4 = 56z^2$, *International Journal of Research Publication and Reviews,* Vol 5 (5), Page – 3224-3226.
https://www.ijrpr.com/current_issues.php

Mahalakshmi, T., Shanthi, J. (2023a). On the positive pell equation $y^2 = 80x^2 + 41$, *International Journal of Research Publication and Reviews,* Volume 4, No 10, 2278-2285.
https://www.ijrpr.com/archive.php?volume=4&issue=10

Mahalakshmi, T., Shanthi, J.(2023b). On the Negative pell equation $y^2 = 80x^2 - 31$, *International Journal of Research Publication and Reviews,* Volume 4, No 10, 2178-2187.
https://www.ijrpr.com/archive.php?volume=4&issue=10

Mahalakshmi, T., Shanthi, J.(2023c). October 2023, On the Positive pell Equation $y^2 = 80x^2 + 4$, *International Journal of Research Publication and Reviews*, Volume 4, No 10, 2188-2195.
https://www.ijrpr.com/archive.php?volume=4&issue=10

Mordell, L.J.(1969). Diophantine equations, *Academic Press*, London.
Diophantine Equations. By L. J. Mordell. Pp. 312. 1969. 90s. (Academic Press, London & New York.) | The Mathematical Gazette | Cambridge Core

Sathiyapriya. R., et al.(2024a), A Glimpse on Homogeneous Ternary Quadratic $3x^2 + 2y^2 = 275z^2$, *Journal of Computational Analysis and Applications,* 33(7),693-696.

https://eudoxuspress.com/index.php/pub/article/view/1130

Sathiyapriya. R., et al.(2024b), Lattice points on the Cone $y^2 + 5x^2 = 86z^2$, *Nanotechnology Perceptions,* 20(7), 890–898.
https://nano-ntp.com/index.php/nano/article/view/3991

Shanthi, J. (2023a). "On Finding Integer Solutions to the Homogeneous Ternary Quadratic Diophantine Equation $2(x^2 + y^2) - 3xy = 32z^2$" *International Journal of Research Publication and Reviews,* Vol 4, no 1, 700-708.
https://www.ijrpr.com/archive.php?volume=4&issue=1

Shanthi, J. (2023b). "Observations on the Hyperbola $x^2 = 20y^2 - 4$" *International Journal of Research Publication and Reviews*, Vol 4, no 1, pp 678-682.
https://www.ijrpr.com/archive.php?volume=4&issue=1

Shanthi, J.Mahalakshmi, T. (2023a). On the Positive pell Equation $y^2 = 87x^2 + 13$, *International Journal of Research Publication and Reviews,* Volume 4, No 10, 2512-2521.
https://www.ijrpr.com/archive.php?volume=4&issue=10

Shanthi, J. Mahalakshmi, T. (2023b). On the Negative pell Equation $y^2 = 87x^2 - 6$, *International Journal of Research Publication and Reviews,* Volume 4, No 10, 2337-2345.
https://www.ijrpr.com/archive.php?volume=4&issue=10

Shanthi, J. Mahalakshmi, T. (2023c). On the Negative pell Equation $y^2 = 87x^2 - 23$, *International Journal of Research Publication and Reviews,* Volume 4, No 10, 2486-2494.
https://www.ijrpr.com/archive.php?volume=4&issue=10

Shanthi, J.,Gopalan.M.A.(2024a). Formulation of special Pythagorean triangles through Integer Solutions of the Hyperbola $y^2 = (k^2 + 2k)x^2 + 1$, *Indian Journal of Science and Technology* ,Volume 17,Issue 41 , 4307-4312.
https://indjst.org/archives?volume=17&issue=41

Shanthi.J, Gopalan. M.A.(2024b). Observations on Non-homogeneous Binary Quadratic Diophantine Equation $3x^2 + 10xy + 4y^2 - 4x + 2y - 7 = 0$, *International Research Journal of Education and Technology* ,Volume 06,Issue 11, 270-276.
https://www.irjweb.com/viewarchives.php?year=2023

Thiruniraiselvi. N., Gopalan. M.A.(2024a).A peer search on Integer Solutions to Ternary Quadratic Equation, *International Journal of Creative Research Thoughts – IJCRT*, Vol 12(7), 107-111.
https://ijcrt.org/track.php?r_id=265988

Thiruniraiselvi. N., Gopalan. M.A.(2024b). On Finding Integer Solutions to Transcendental Equation with Six Unknowns $\sqrt[3]{X+Y}+\sqrt{xy}=z^{2}+w^{2}$, *Indian Journal of Science and Technology*, Vol. 17(19), 2992-3001.
https://indjst.org/articles/a-portrayal-of-integer-solutions-to-transcendental-equation-with-six-unknowns-

Vidhyalakshmi. S. et al.(2022a). On The Second Degree Equation With Three Unknowns $5x^{2}+4y^{2}=189z^{2}$, *International Journal of Multidisciplinary Research and Growth Evaluation* , Vol 03(03), 197-204.
https://www.allmultidisciplinaryjournal.com/archives/year-2022.vol-3.issue-3?page=4

Vidhyalakshmi. S., Gopalan. M.A.(2021b). Observations on the Surd Equation $\sqrt{2z-4}=\sqrt{x+\sqrt{m^{2}+4}\,y}+\sqrt{x-\sqrt{m^{2}+4}\,y}$, *International Journal of Research Publication and Reviews*, Vol. 2(11),437-441.
Observations on the Surd Equation (ijrpr.com)